Three Points Make a Triangle… Or a Circle

Peter Schröder

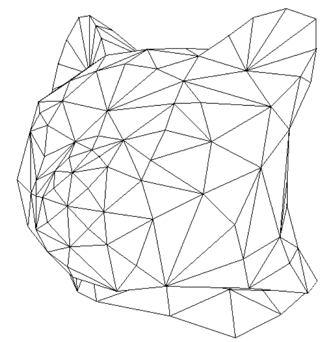
joint work with

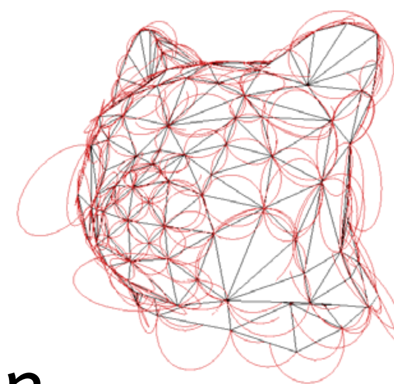
Liliya Kharevych, Boris Springborn, Alexander Bobenko

In This Section

Circles as basic primitive

- it's all about the underlying geometry!
  - Euclidean: triangles
  - conformal: circles
- two examples
  - conformal parameterization
  - discrete curvature energies

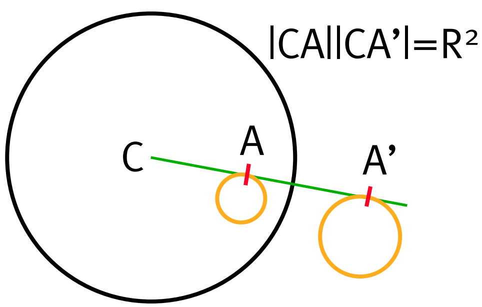




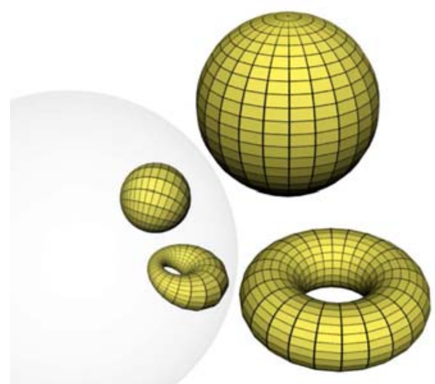
Geometries

The Erlangen program (1872)

- geometry through symmetries
  - affine, perspective
  - conformal



Möbius xform

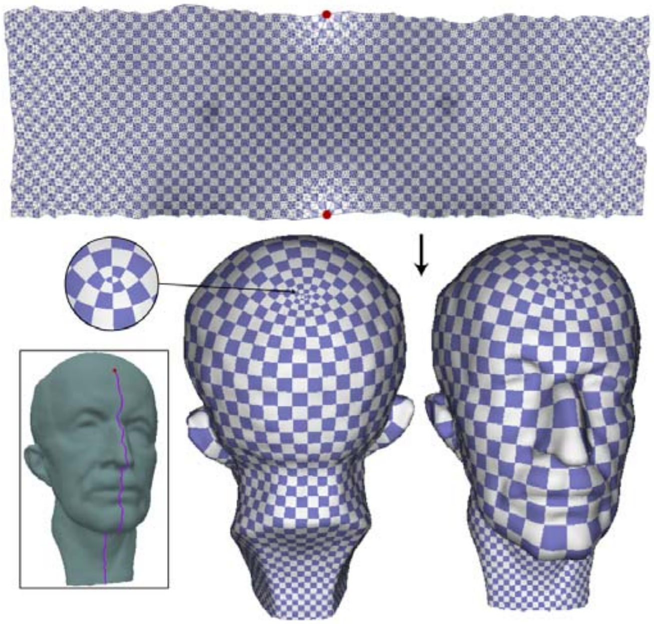



Library of Congress

Conformal Mappings

Piecewise Linear Surfaces

- map to domain
- discrete conformal
- preserve angles
  - as well as possible
- circles as primitives

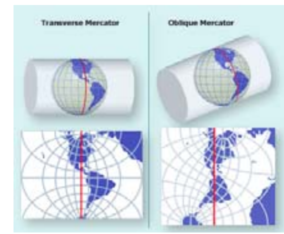


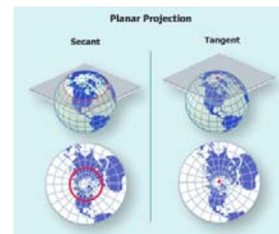
PARAMATERIZATIONS

An old problem...

- can't have it all
  - keep angles
  - keep areas

Mercator (1512-1594);
Frontispiece of *Atlas Sive Cosmographicae* (1585-1595)





USGS Map Projections Site

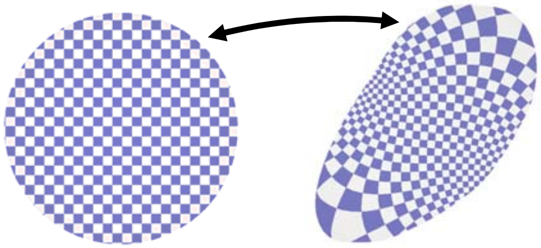
Our setup

- find discrete conformal map from triangle mesh to Euclidean domain

Conformal Maps

Mathematical basis

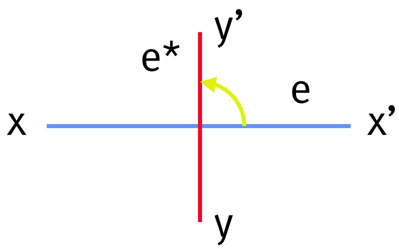
- Riemann mapping theorem
  - unique (up to Möbius xforms)



$$f : \Omega_1 \subset \mathbb{C} \rightarrow \Omega_2 \subset \mathbb{C}$$

$$\boxed{i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}}$$

$$f_x \perp f_y \qquad |f_x| = |f_y|$$

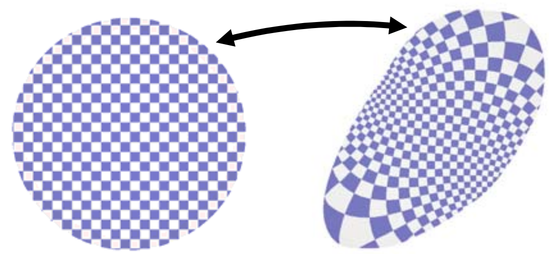


$$\frac{f(y')-f(y)}{l(y,y')} = i\frac{f(x')-f(x)}{l(x,x')}$$

Conformal Maps

Mathematical basis

- Riemann mapping theorem
  - unique (up to Möbius xforms)



$$f : \Omega_1 \subset \mathbb{C} \to \Omega_2 \subset \mathbb{C}$$

$$\boxed{i\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}}$$

$$0 = \sum_{e_{ij}} (\cot \alpha_{ij} + \cot \alpha_{ji})(v_i - v_j)$$

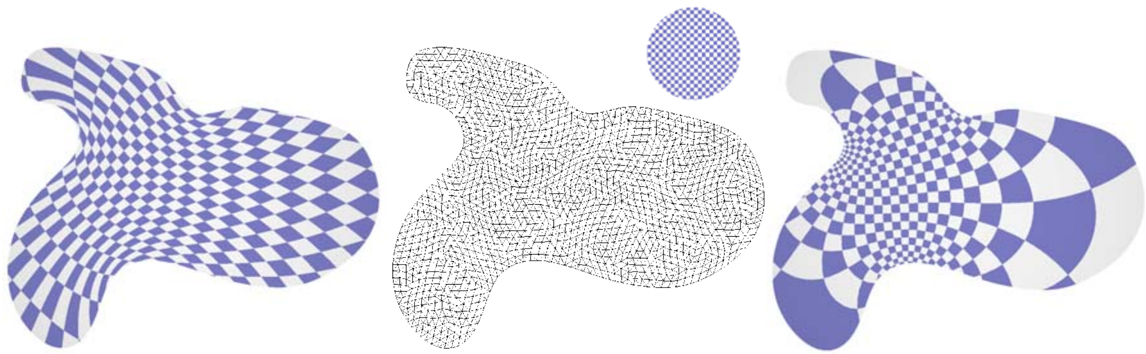
Intrinsic angles in original mesh

Texture coordinates

Sounds Great!

You knew there was a catch...

- Laplace problem
  - Dirichlet or Neumann bndry. cond.

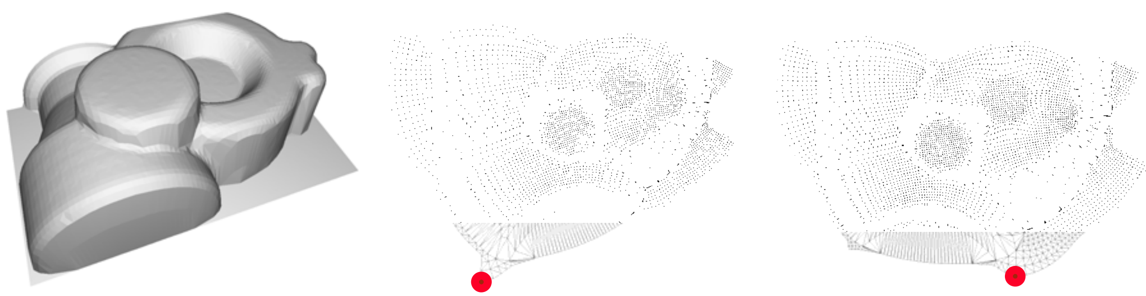


$$\Delta u = 0$$
$$u|_{\partial\Omega} = u_0$$

Sounds Great!

You knew there was a catch...

- Laplace problem
  - Dirichlet or Neumann bndry. cond.



- too much control or too little...

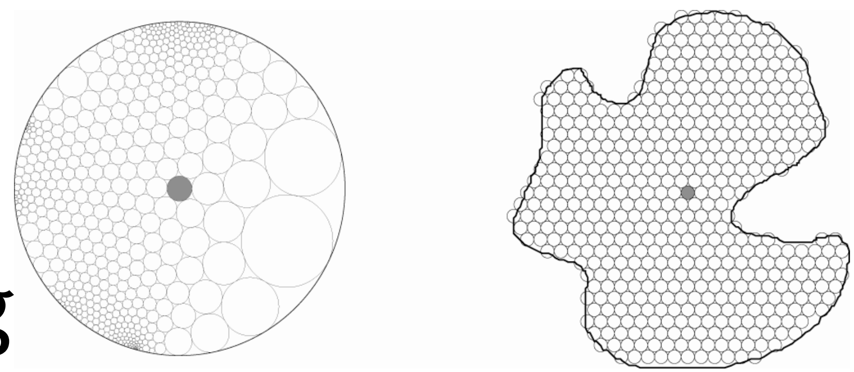
An Old Idea

Riemann mapping theorem

- conformal maps map infinitesimal circles to infinitesimal circles

Thurston (85)

- finite circles
- circle packing



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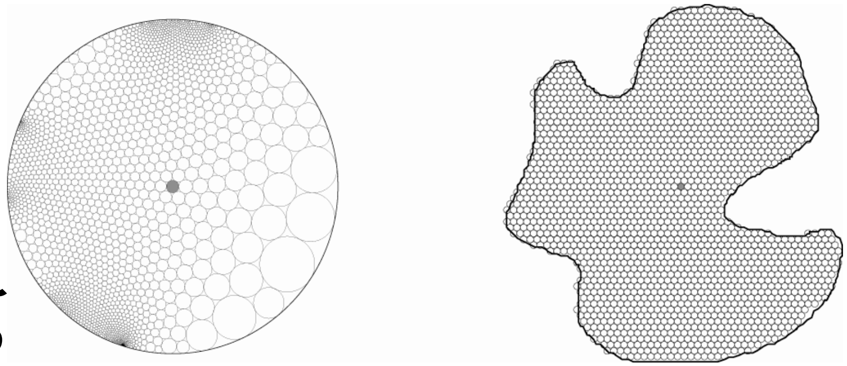

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An Old Idea

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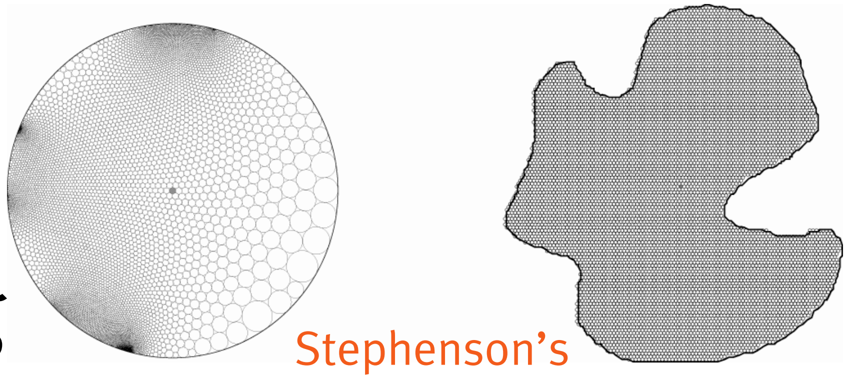

Stephenson's CirclePack

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History

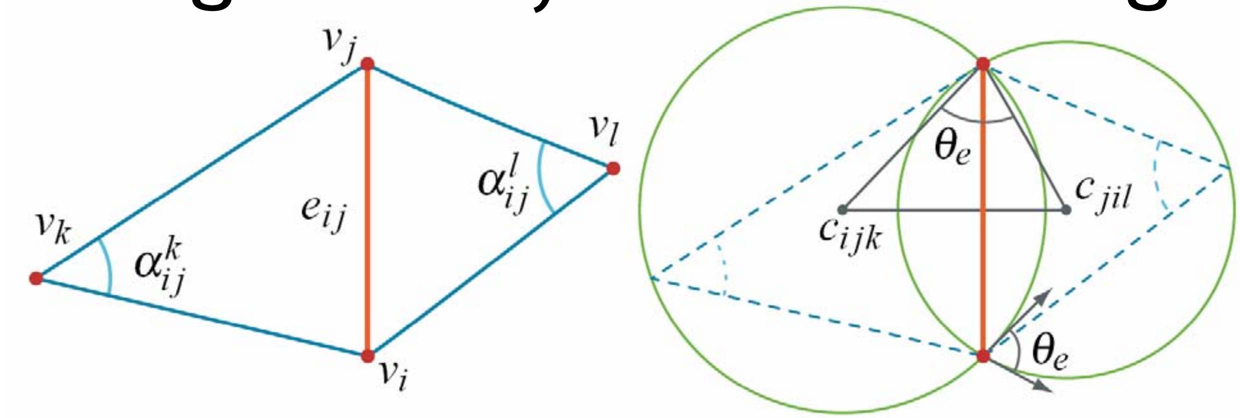
Theory

- early: Koebe 36; Andreev 70
- modern: Rudin & Sullivan 87 (hex packing); He & Schramm 98 (C^∞ convgce.)
- variational approaches 91-02
  - de Verdière, Brägger, Rivin, Leibon, Bobenko & Springborn

Basic Setup

Given a mesh

- local geometry around an edge



$$\forall e_{ij} \in E : \theta_e = \begin{cases} \pi - \alpha_{ij}^k - \alpha_{ij}^l \\ \pi - \alpha_{ij}^k \end{cases}$$

Circle Pattern Problem

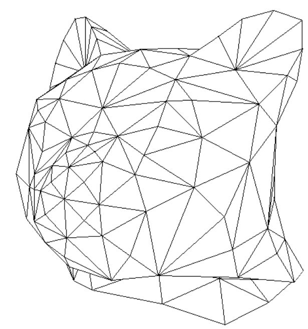
Rivin 94, Bobenko & Springborn 04

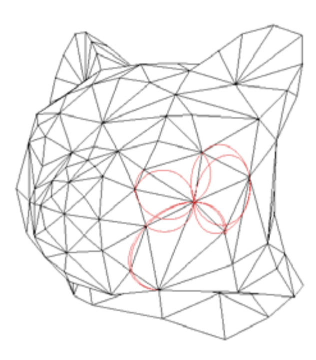
- given a triangulation K
- an angle assignment

  $\forall e_{ij} \in E : 0 < \theta_e < \pi$

- sum conditions

  $\forall v_i \in V_{\text{int}} : \sum_{e \ni v_i} \theta_e = 2\pi$

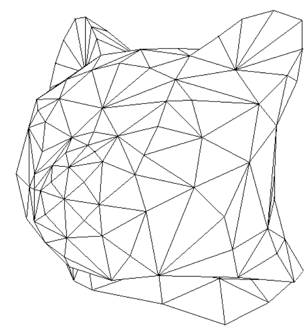




Circle Pattern Problem

Rivin 94, Bobenko & Springborn 04

- given a triangulation K



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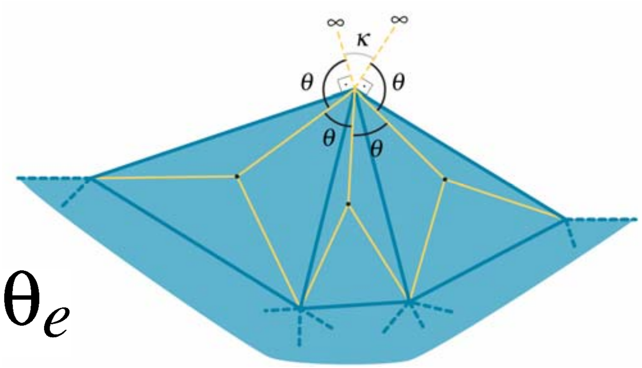


$$\forall v_i \in V_{\mathrm{int}} : \sum_{e \ni v_i} \theta_e = 2\pi$$

$$\forall v_i \in V_{\mathrm{bdy}} : \kappa_i = 2\pi - \sum_{e \ni v_i} \theta_e$$

Circle Pattern Problem

Uniquely realizable iff

- a coherent angle system exists

$$\hat{\alpha}_{ij}^{k} > 0 \qquad \forall t_{ijk} \in T : \hat{\alpha}_{ij}^{k} + \hat{\alpha}_{jk}^{i} + \hat{\alpha}_{ki}^{j} = \pi,$$

$$\forall e_{ij} \in E : \theta_e = \begin{cases} \pi - \hat{\alpha}_{ij}^{k} - \hat{\alpha}_{ij}^{l} \\ \pi - \hat{\alpha}_{ij}^{k} \end{cases}$$

- linear feasibility problem

Geometry at an Edge

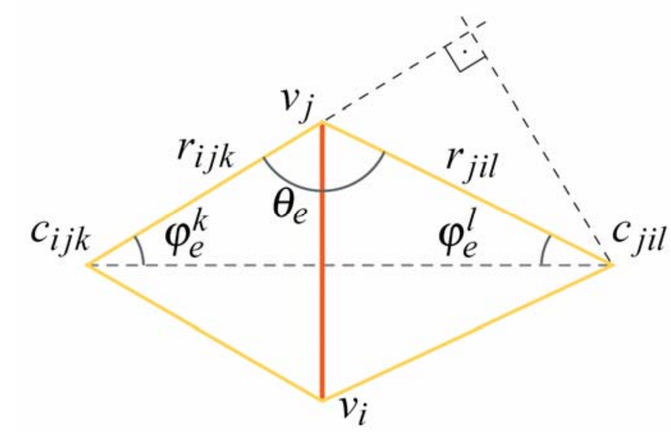
Relationship between variables

- angle and radii

$x = \log r_{ijk} - \log r_{jil}$

$$\varphi_e^k = \begin{cases} f_e(x) = \tan^{-1} \dfrac{\sin\theta_e}{e^x - \cos\theta_e} & e \in E_{\mathrm{int}} \\ \pi - \theta_e & e \in E_{\mathrm{bdy}} \end{cases}$$

$$\boxed{\forall t \in T : 2\pi = \sum_{e \in t} 2\varphi_e^t}$$



Energy

Solution is unique minimum!

- convex energy in $\rho_{ijk} = \log r_{ijk}$

$$\boxed{\begin{aligned} S(\rho) = \sum_{e \in E_{\mathrm{int}}} \Big(&\mathrm{ImLi}(e^{\rho_k - \rho_l + i\theta_e}) + \mathrm{ImLi}(e^{\rho_l - \rho_k + i\theta_e}) \\ &- (\pi - \theta_e)(\rho_k + \rho_l) \Big) \\ - \sum_{e \in E_{\mathrm{bdy}}} &2(\pi - \theta_e)\rho_k + 2\pi \sum_{t \in T} \rho_t \end{aligned}}$$

- easy gradients and Hessians!

Algorithm

Angle assignment

- quadratic program
- boundary curvatures
  - free, prescribed

Minimize energy

Lay out circles

- angles and radii determine layout

$$Q(\hat{\alpha}) = \sum |\hat{\alpha}_{ij}^k - \alpha_{ij}^k|^2$$

$$\forall \hat{\alpha}_{ij}^k : \hat{\alpha}_{ij}^k > 0$$

$$\forall e_{ij} \in E_{\text{int}} : \hat{\alpha}_{ij}^k + \hat{\alpha}_{ij}^l < \pi$$

$$\forall t_{ijk} \in T : \hat{\alpha}_{ij}^k + \hat{\alpha}_{jk}^i + \hat{\alpha}_{ki}^j = \pi$$

$$\forall v_k \in V_{\text{int}} : \sum_{t_{ijk} \ni v_k} \hat{\alpha}_{ij}^k = 2\pi$$

$$\forall v_k \in V_{\text{bdy}} : \sum_{t_{ijk} \ni v_k} \hat{\alpha}_{ij}^k < 2\pi$$

$$\forall v_k \in V_{\text{bdy}} : \sum_{t_{ijk} \ni v_k} \hat{\alpha}_{ij}^k = \pi - \kappa_k$$

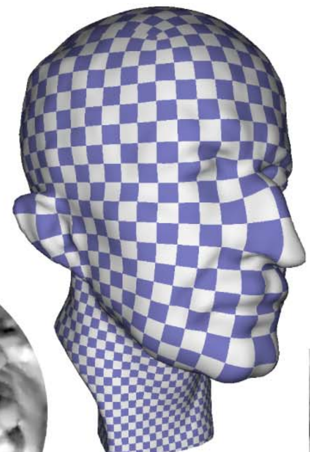
RESULTS

Disk boundary

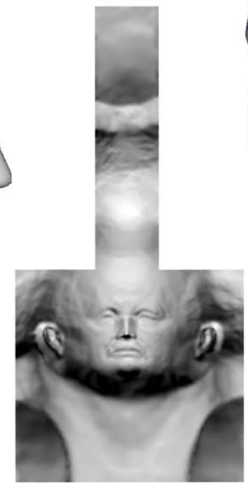
39.6k F

28+8s

Prescribed boundary



37.9k F



26+9s

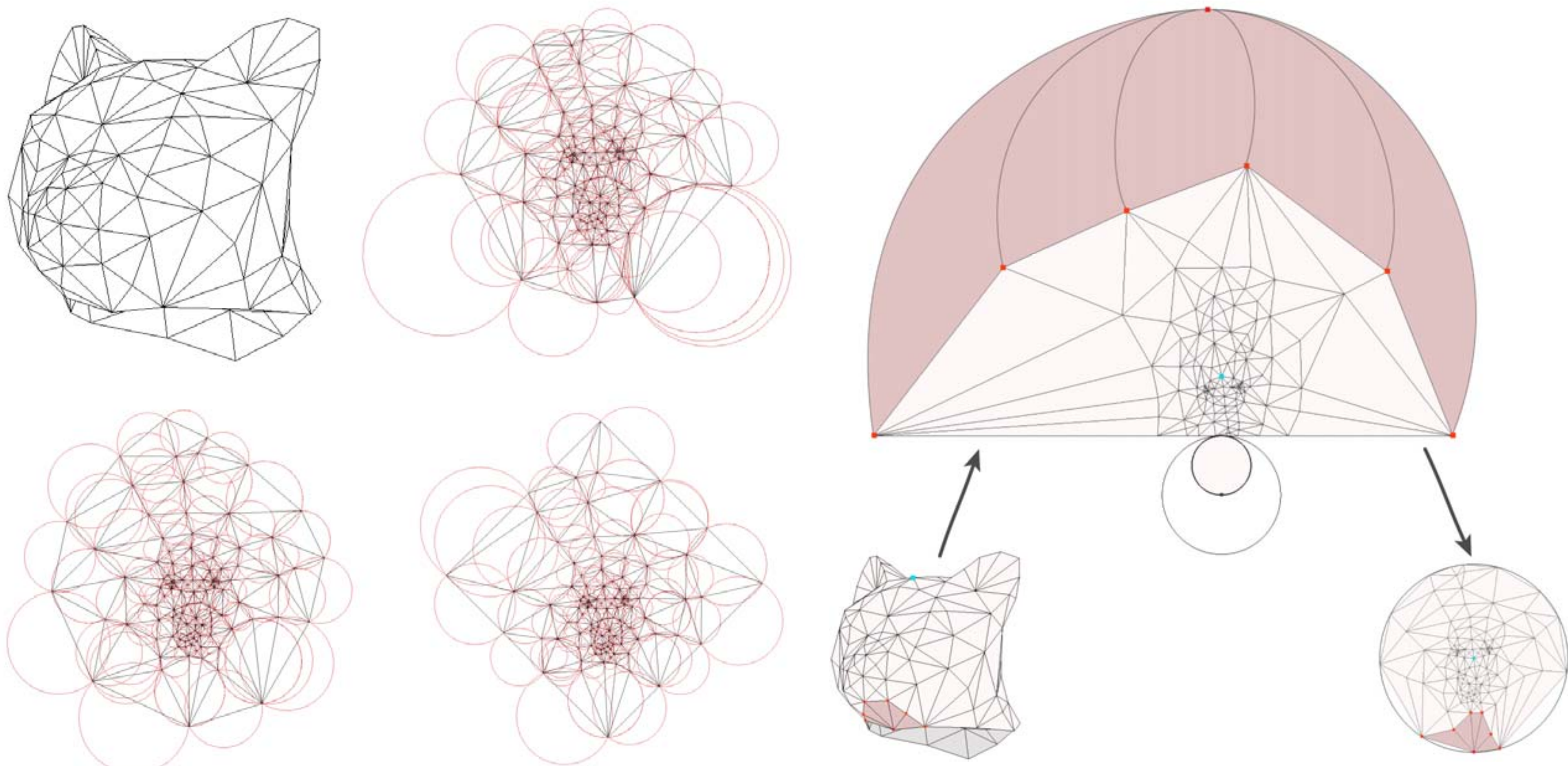
Free boundary

12.6k F

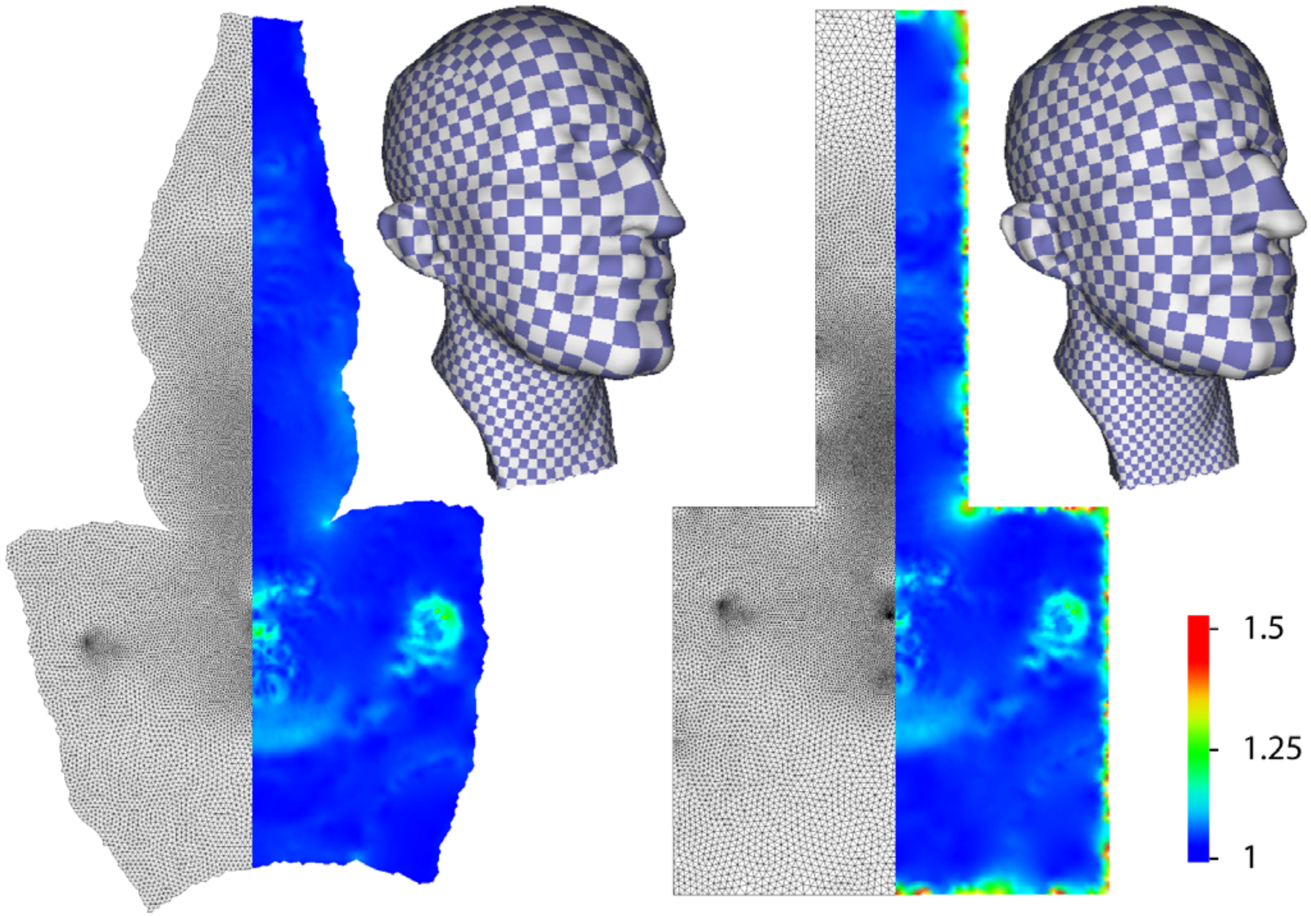
7+2s

Boundary Control

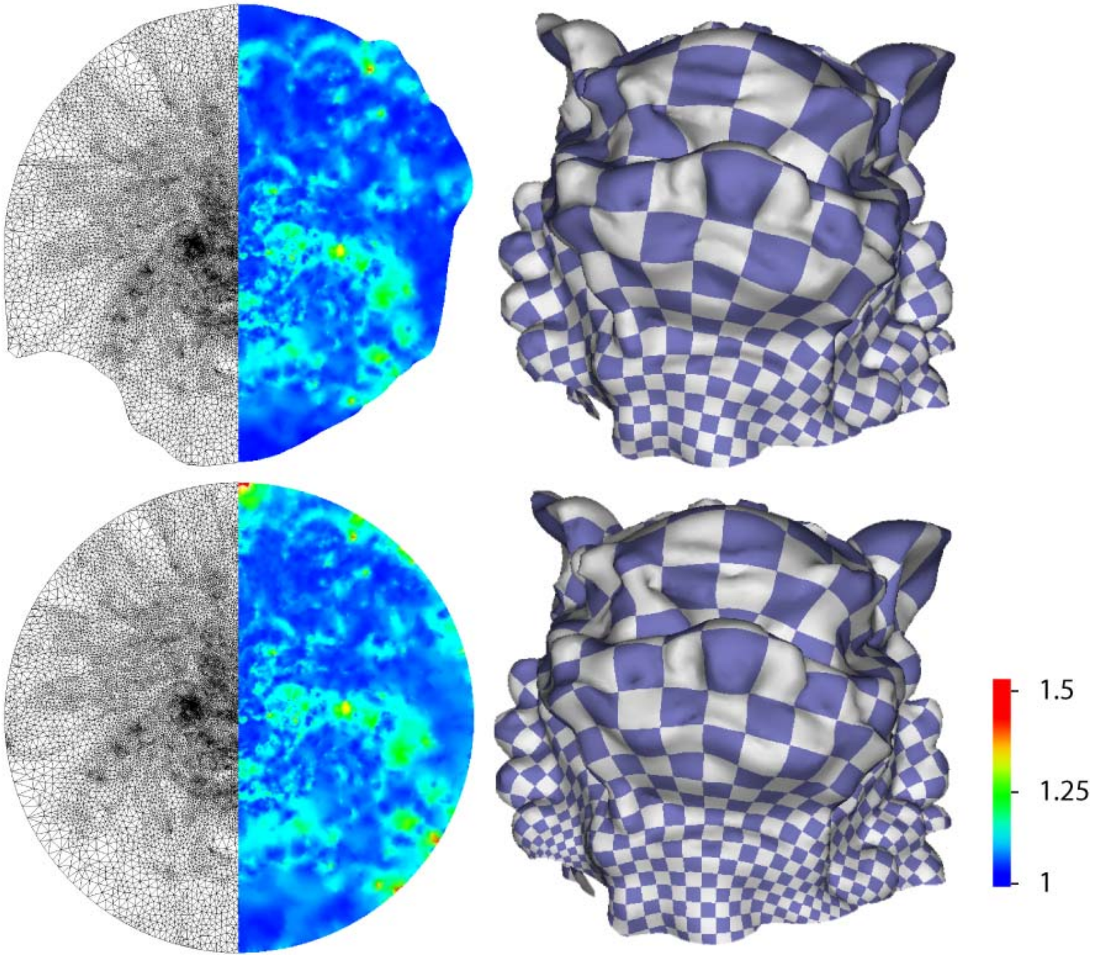
You get to control curvature...



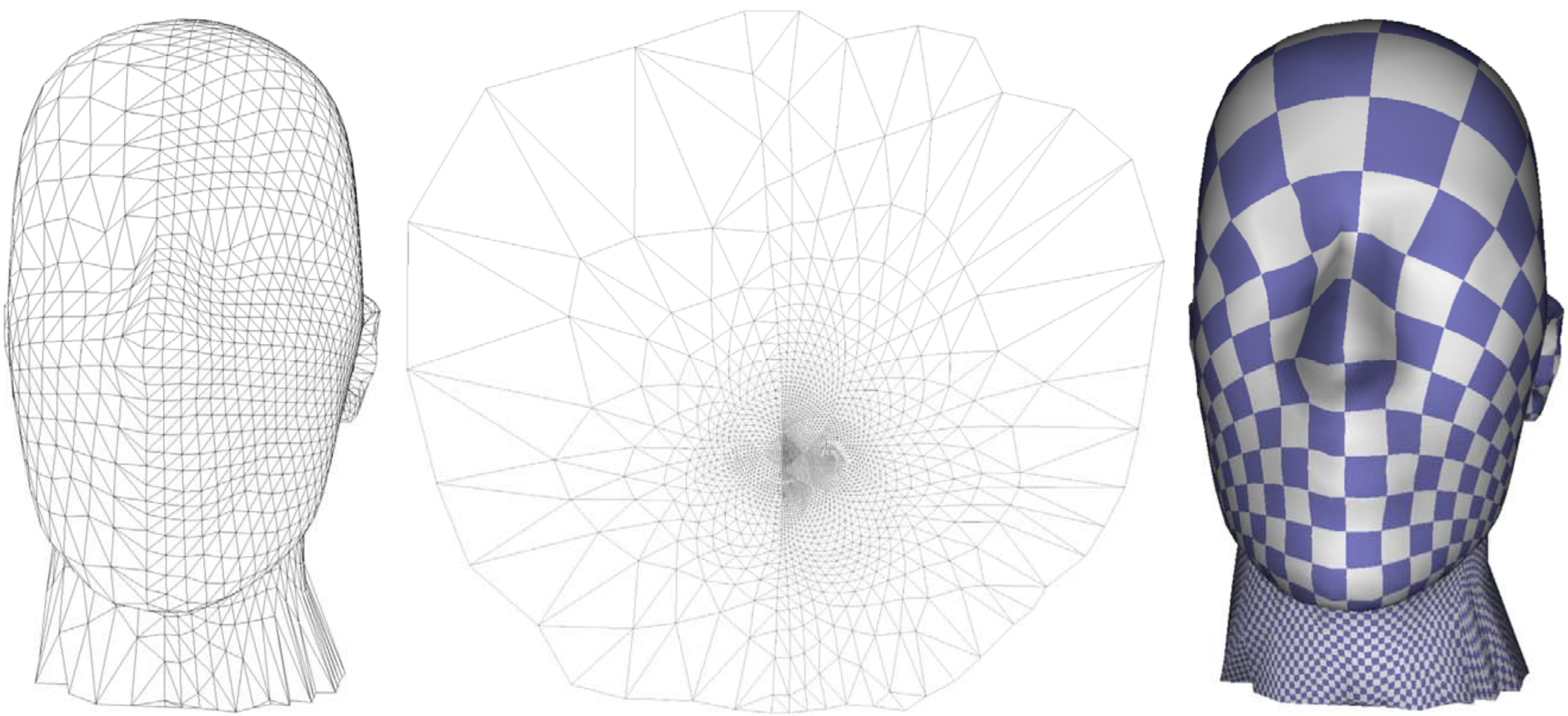
Quasi-Conformal Distr.



Quasi-Conformal Distr.



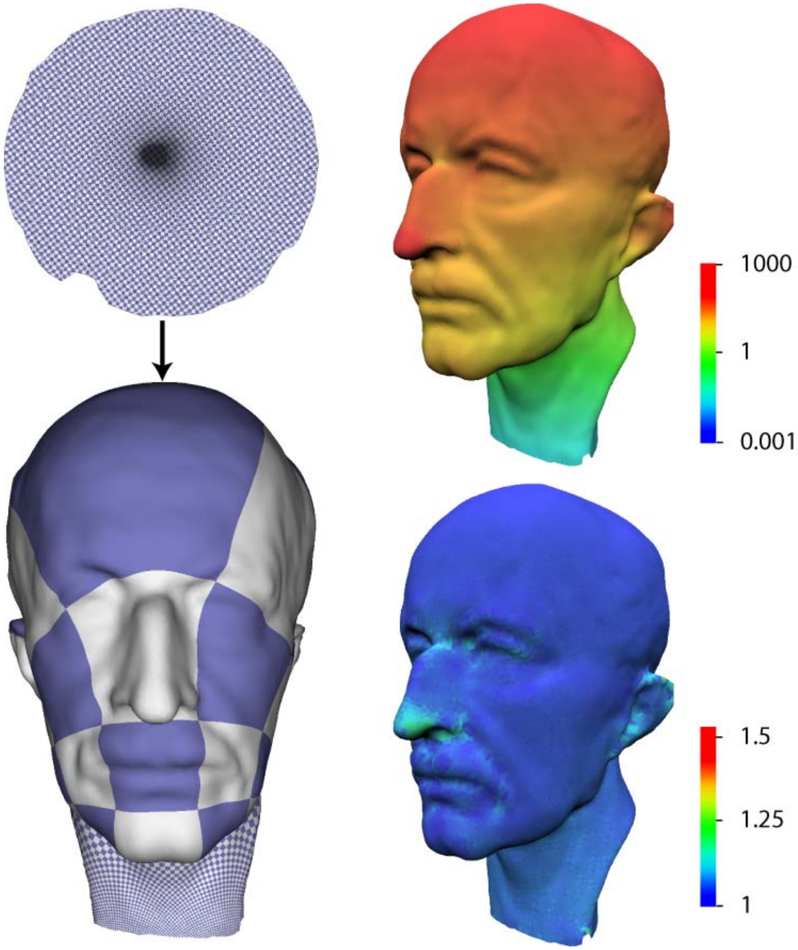
Robustness



Problems

The price to pay

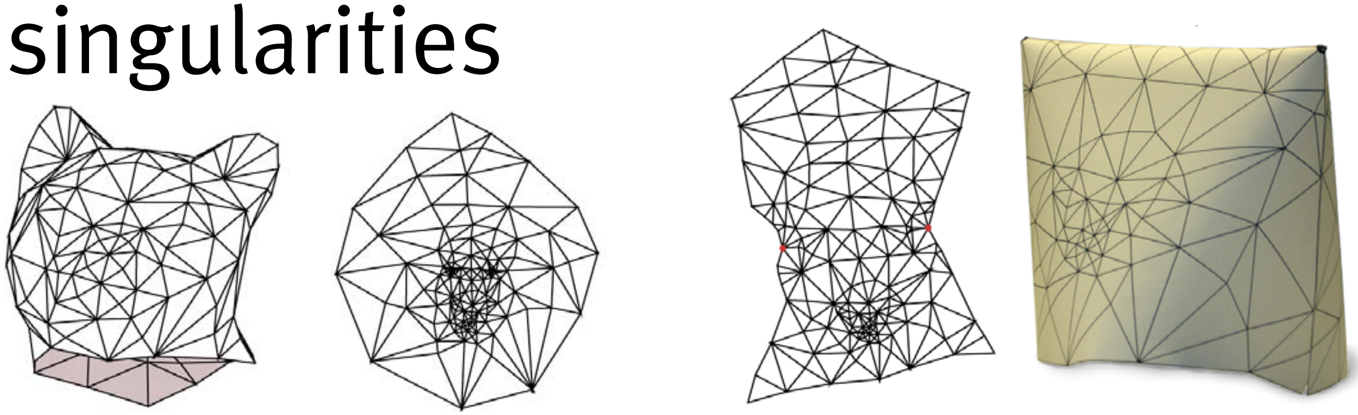
- want angles (nearly) preserved
- must suffer large area distortion



Piecewise Flat

Back to first principles

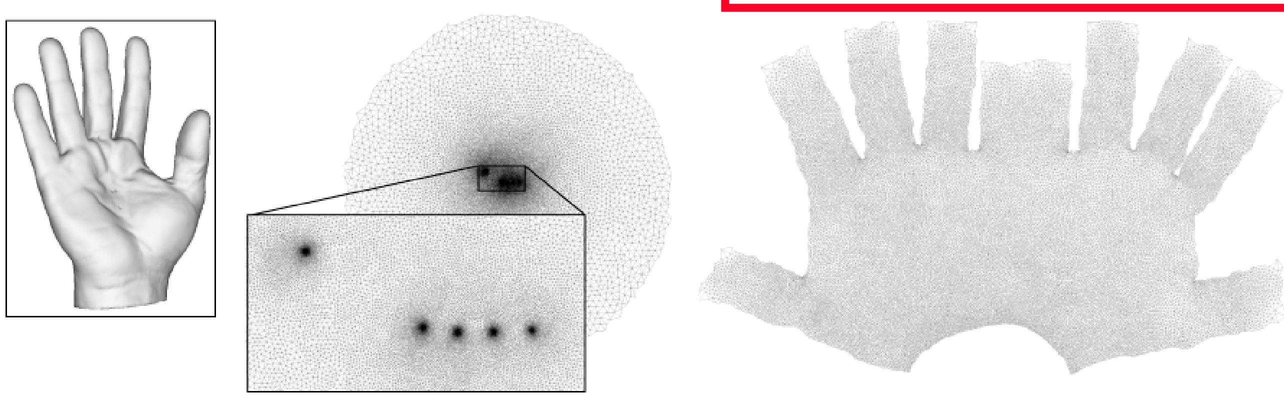
- what does the mesh give us?
  - everywhere flat with some exceptions
- Euclidean metric with cone singularities



Cone Singularities

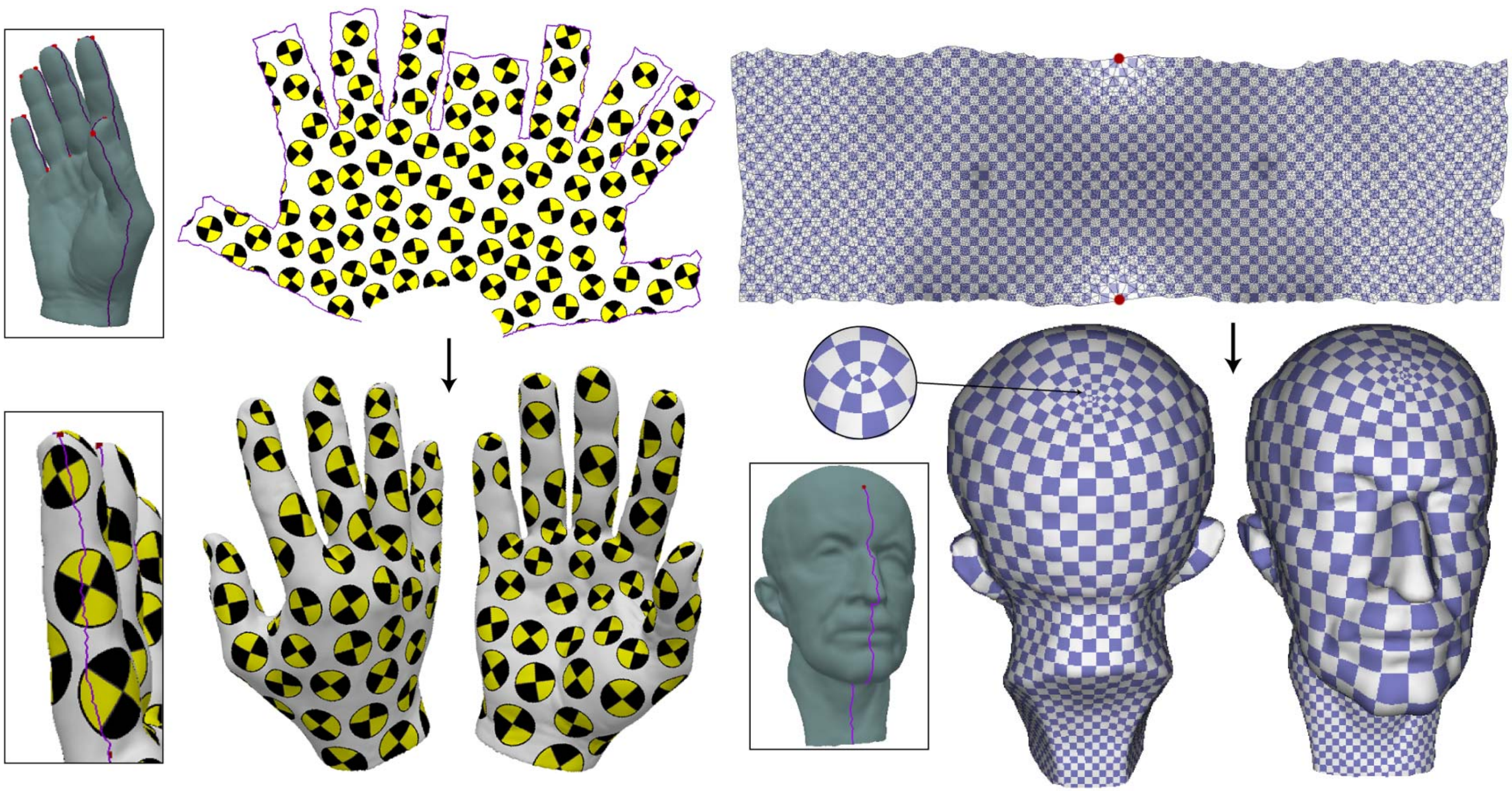
Circle pattern approach

- allows for cone singularities!
- set cone vertices $\boxed{2\pi \neq \Theta_i = \sum_{e_{ij} \ni v_i} \theta_{ij}}$

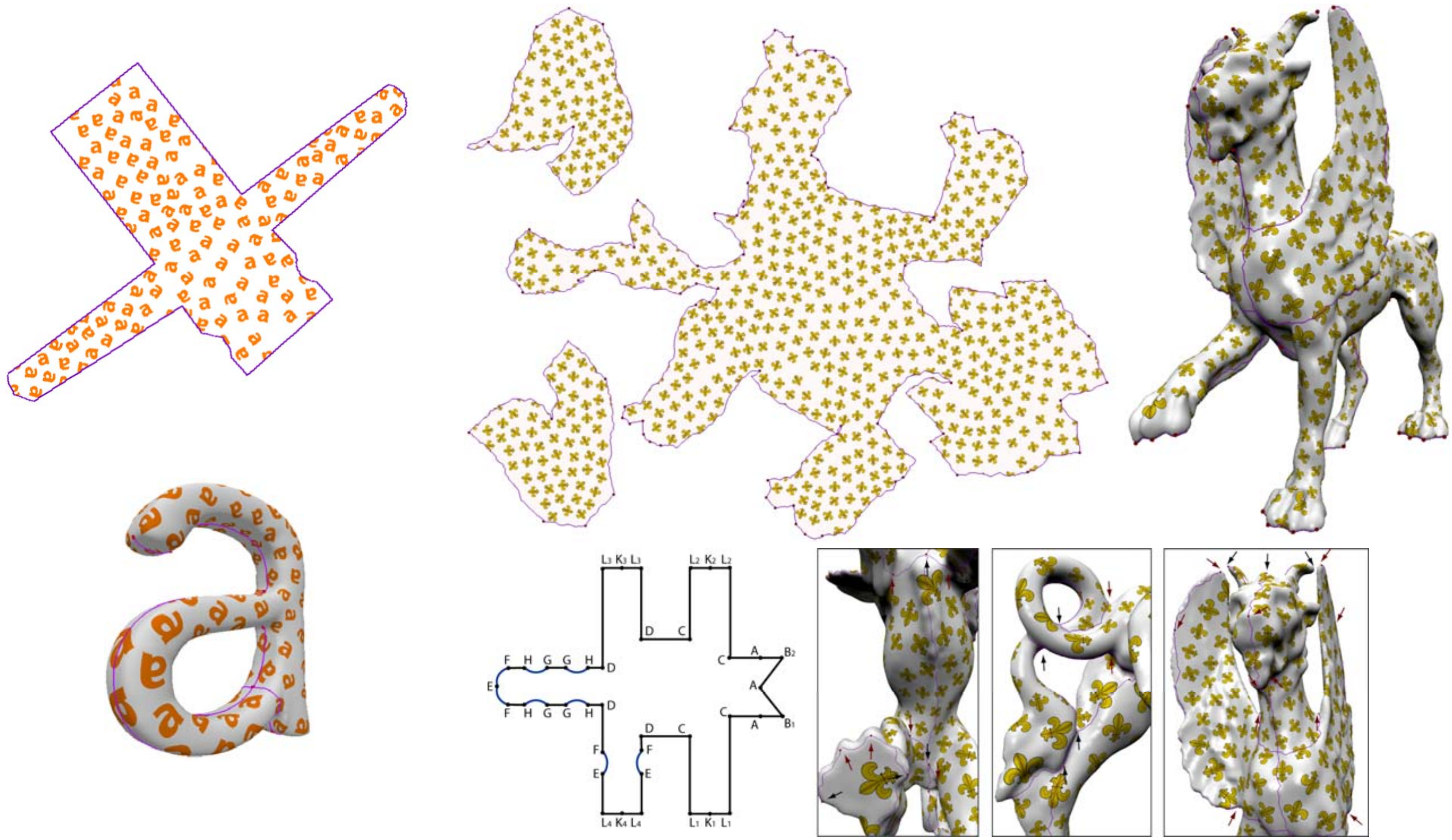


- rest of machinery works as before

Examples



Examples



Properties

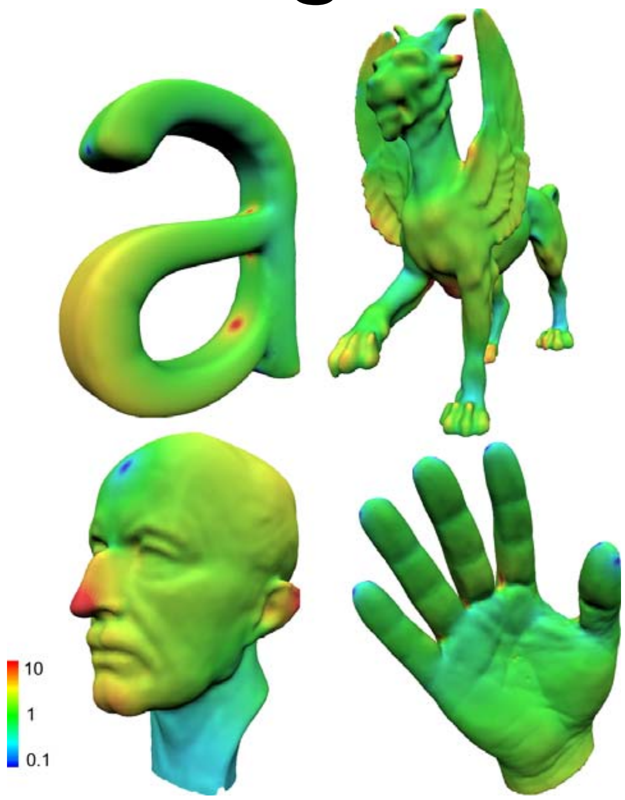
Circle patterns with cone sing.

- discrete conformal

PROPERTIES

Circle patterns with cone sing.

- discrete conformal
- low area distortion
- arbitrary topology
- no cutting a priori!
- globally continuous



Summary

Discrete conformal mappings

- formulate as circle pattern problem
- solution is min. of convex energy
  - simple gradient and Hessian
- cone singularities
  - no cutting a priori & arbitrary topology

You *can* have both: area & angle!

More Fun with Circles

Willmore energy of a surface

vanishes iff surface is sphere

$$\boxed{E_W(S) = \int_S ((H/2)^2 - K)\, dA}$$

$$\int_S (\kappa_1 - \kappa_2)^2\, dA$$

More Fun with Circles

Willmore energy of a surface

$$\boxed{E_W(S) = \int_S ((H/2)^2 - K)\, dA}$$

Möbius invariant

- of interest: minimizers
  - theory of surfaces

Conformal Geometry

More Fun with Circles

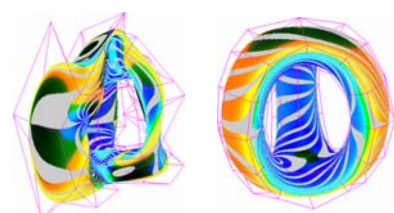
Willmore energy of a surface

$$\boxed{E_W(S) = \int_S ((H/2)^2 - K)\, dA}$$

$$\boxed{\int_S {\kappa_1}^2 + {\kappa_2}^2\, dA}$$

- of interest: minimizers
  - theory of surfaces
  - geometric modeling

Conformal Geometry



More Fun with Circles

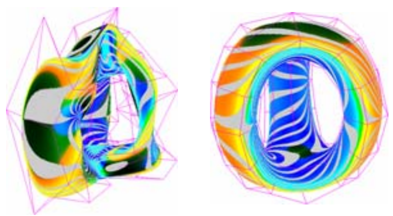
Willmore energy of a surface

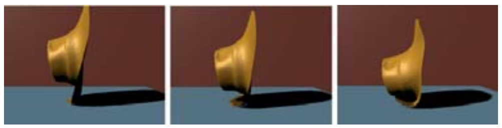
$$\boxed{E_W(S) = \int_S ((H/2)^2 - K)\, dA}$$

$$\boxed{\int_S \alpha + \beta(H - H_0)^2\, dA}$$

- of interest: minimizers
  - theory of surfaces
  - geometric modeling
  - physical modeling

Conformal Geometry

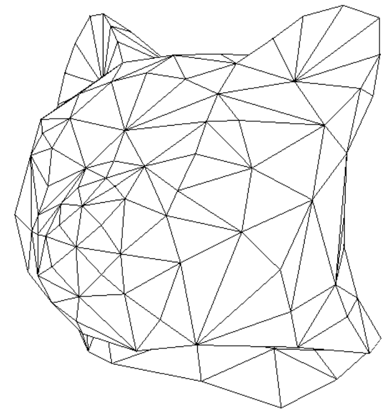


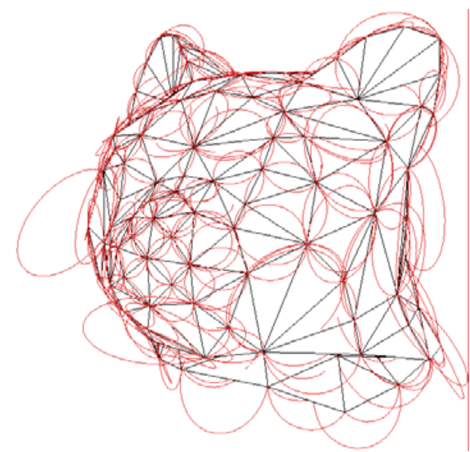


Discrete Willmore Flow

Approach

- simplicial 2-manifolds
- preserve symmetries
  - Möbius invariance
  - non-linear formulation
- semi-implicit integration
- boundary conditions





Previous Work

Discrete setting

- Discrete Differential Geometry
  - 4th order flows [SK01] [XPB05] [YB02]
  - use existing lower order operators
- discretized continuous setting
  - level set [TWBO03] [DR04]
  - FEM [DDE03] [HGR01] [CDD*04]

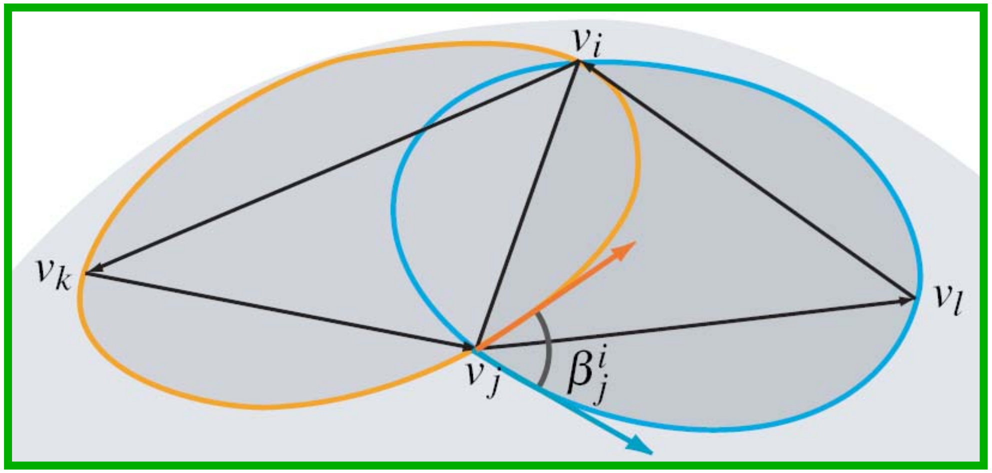
Disc. Willmore Energy

Definition [B05]

- object of conformal geometry...
  - ...use circles and angles

$$W_i = \sum_{e_{ij}} \beta^i_j - 2\pi$$

at each vertex

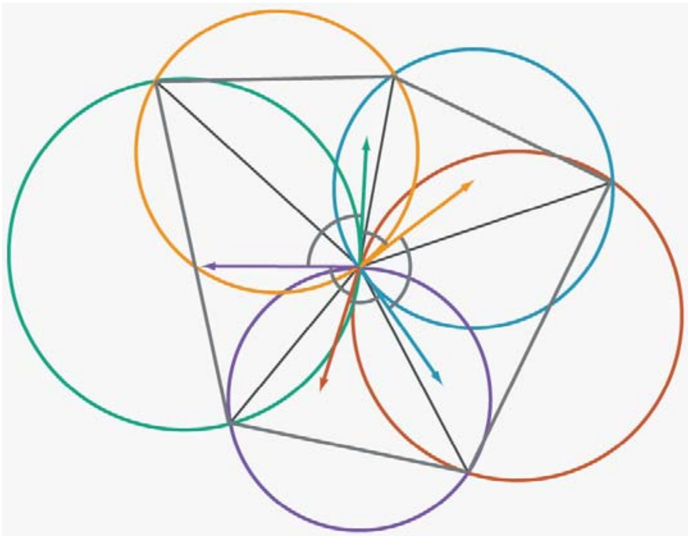


Properties I

Discrete Willmore energy

- vanishes iff co-spherical & convex

$$\boxed{\sum_j \beta_j \geq 2\pi}$$



Properties I

Discrete Willmore energy

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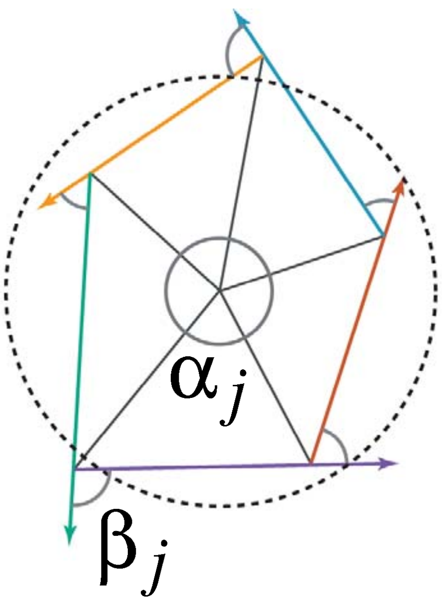
$$\boxed{\sum_j \beta_j \geq 2\pi}$$

$$\boxed{\sum_j \alpha_j \leq \sum_j \beta_j}$$



Properties I

Discrete Willmore energy

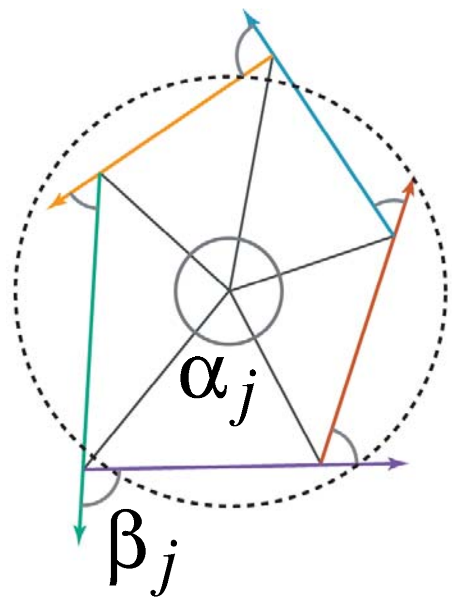
- vanishes iff co-spherical & convex

$$\boxed{\sum_j \beta_j \geq 2\pi} \qquad \boxed{\sum_j \alpha_j \leq \sum_j \beta_j}$$

$$\boxed{\textstyle\sum_j \beta_j - 2\pi} \qquad \boxed{2\pi - \textstyle\sum_j \alpha_j}$$

$$\boxed{W_i + K_i \geq 0}$$

$$\boxed{H^2\, dA \geq 0}$$



Properties II

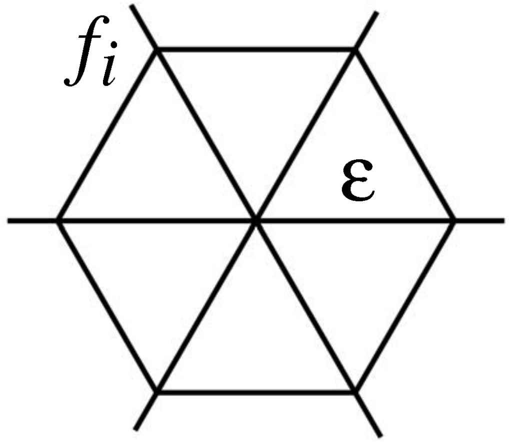
Discrete Willmore energy

- smooth limit

$$R = \lim_{\varepsilon \to 0} \frac{W(D_\varepsilon)}{\mathcal{W}(D_\varepsilon)}$$

discrete

continuous



- independent of $\kappa_{1,2}$

Properties II

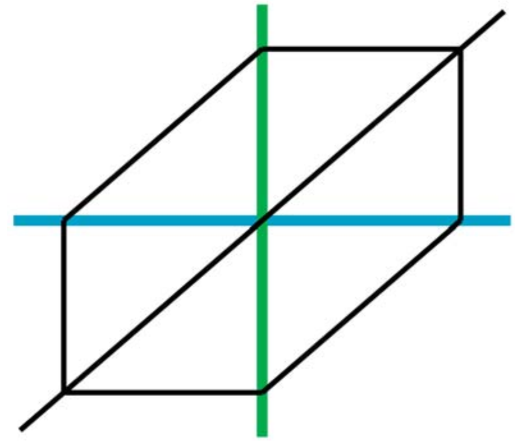
Discrete Willmore energy

- smooth limit

discrete

$$R = \lim_{\varepsilon \to 0} \frac{W(D_\varepsilon)}{\mathcal{W}(D_\varepsilon)}$$

continuous

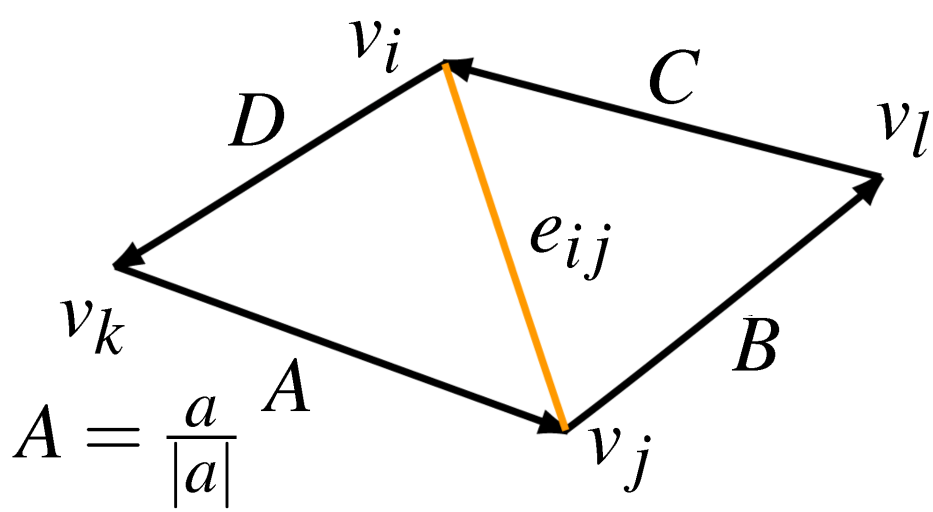


- independent of $\kappa_{1,2}$
- $R \geq 1$ with equality for curvature line parameterizations

Properties III

Evaluation

- symmetry

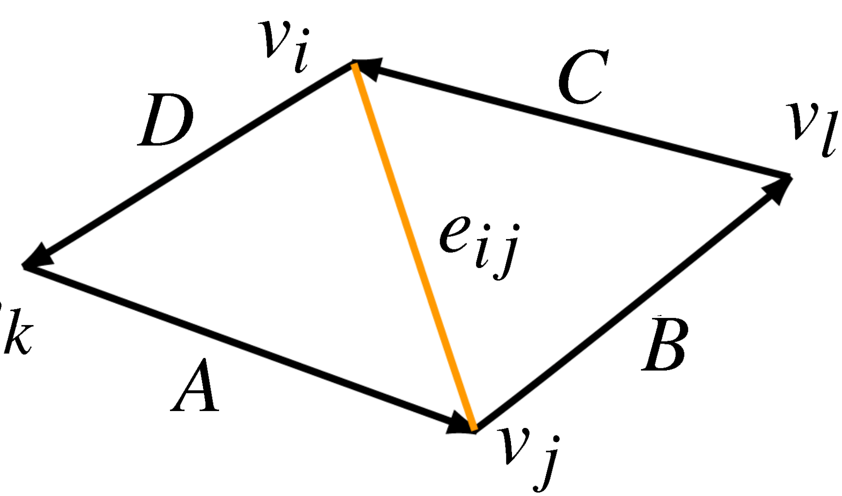


$$\cos\beta^i_j = \langle A, C\rangle\langle B, D\rangle - \langle A, B\rangle\langle C, D\rangle - \langle B, C\rangle\langle D, A\rangle$$

Properties III

Evaluation

- symmetry
    - derivatives
- gradient
    - N x N linear system in (a,b,c,d)



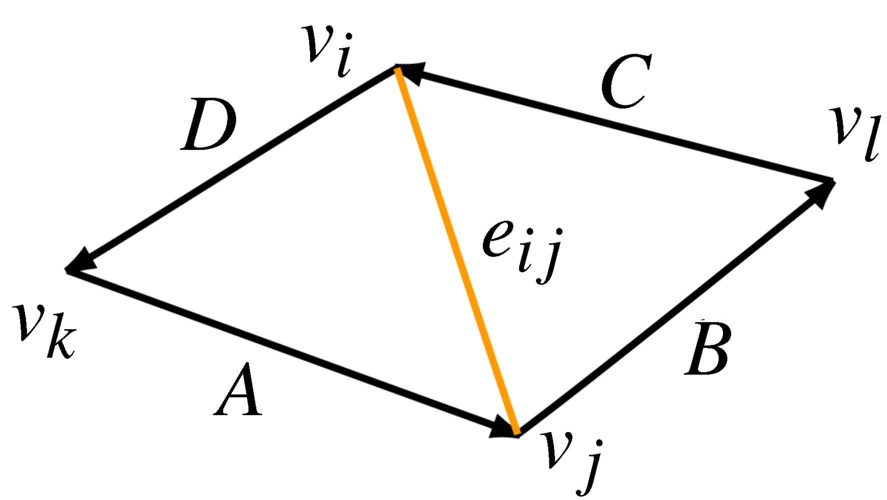
$$\frac{\cos\beta}{|a|^2} - \frac{\langle B, C\rangle}{|a||d|}$$

$$\frac{d}{dv_k}\beta = -\csc(\beta)(\gamma_a a + \gamma_b b - \gamma_c c - \gamma_d d)$$

Properties III

Evaluation

- symmetry
  - derivatives
- gradient
  - N x N linear system in (a,b,c,d)



$$\frac{\langle A, C\rangle}{|b||d|} + \frac{\langle C, D\rangle}{|b||a|}$$

$$\boxed{\frac{d}{dv_k}\beta = -\csc(\beta)(\gamma_a a + \gamma_b b - \gamma_c c - \gamma_d d)}$$

Properties III

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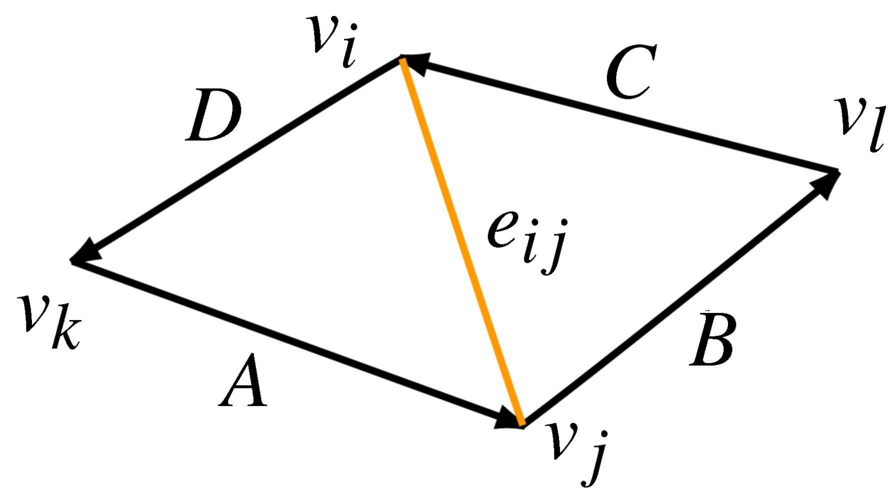

$$\frac{\langle A,B\rangle}{|c||d|} + \frac{\langle B,D\rangle}{|c||a|}$$

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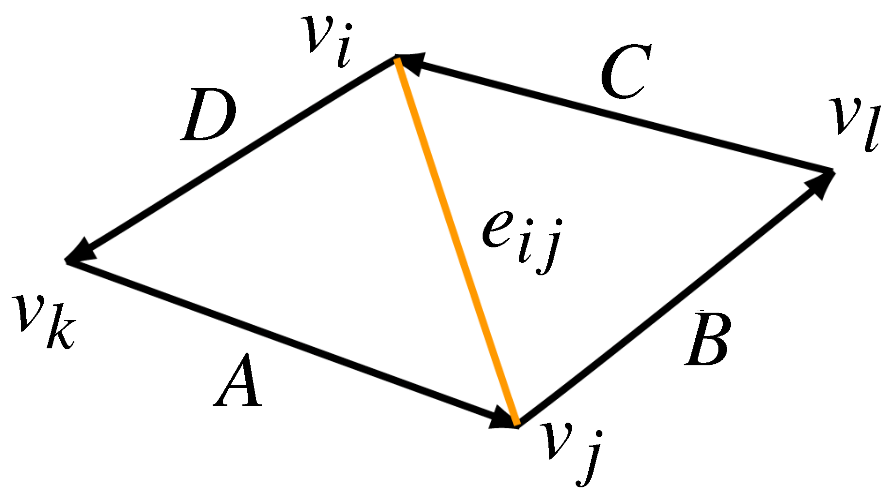


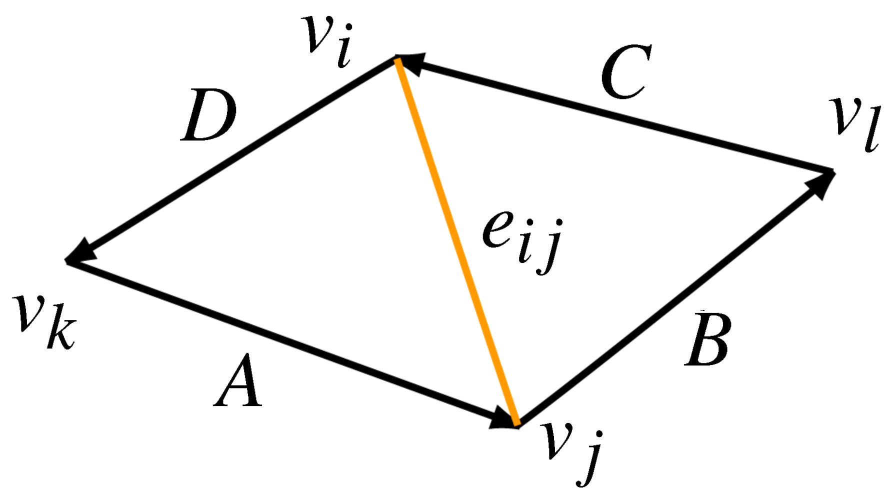
$$\frac{\cos\beta}{|d|^2} - \frac{\langle B,C\rangle}{|a||d|}$$

$$\frac{d}{dv_k}\beta = -\csc(\beta)(\gamma_a a + \gamma_b b - \gamma_c c - \gamma_d d)$$

PROPERTIES III

Evaluation

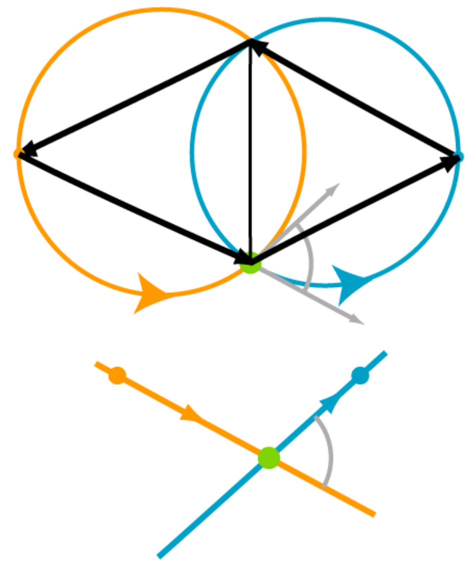
- symmetry
  - derivatives
- gradient
  - N x N linear system in (a,b,c,d)
  - exact Hessian is 3N x 3N...



$$\frac{d}{dv_k}\beta = -\csc(\beta)(\gamma_a a + \gamma_b b - \gamma_c c - \gamma_d d)$$

Properties IV

- Gradient singularity
  - what about csc(β)?
  - direction of decrease
- Boundary conditions
  - fixed: easy



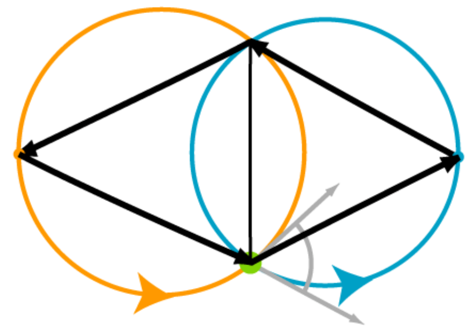
Properties IV

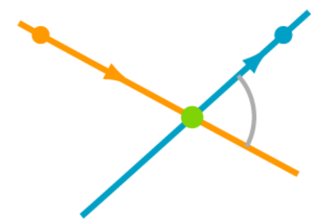
Gradient singularity

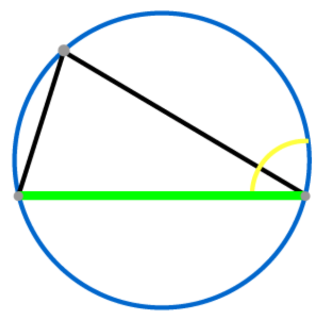
- what about csc(β)?
- direction of decrease

Boundary conditions

- fixed: easy
- free: add vertex at ∞
  - boundary edge β







Properties IV

Gradient singularity

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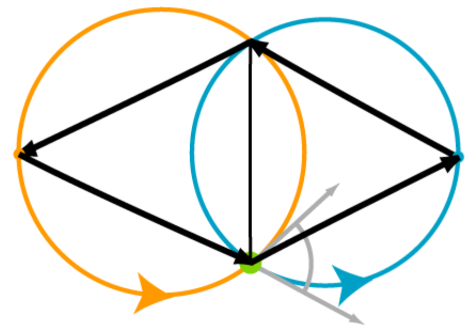

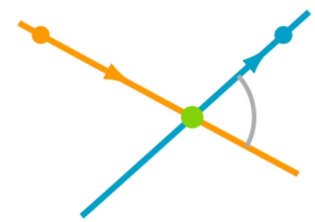

Boundary conditions

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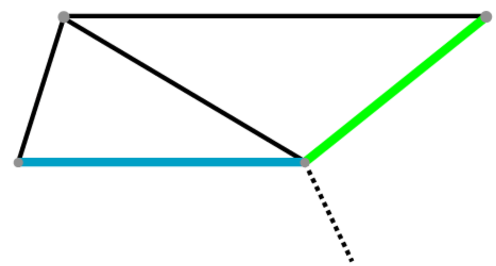

PROPERTIES IV

Gradient singularity

- what about csc(β)?
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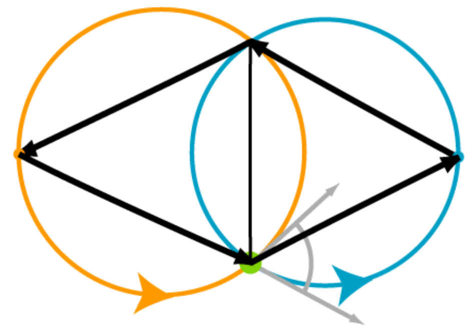

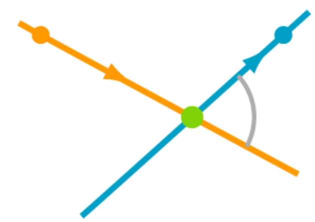

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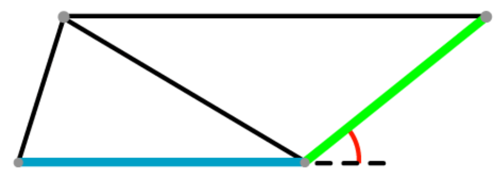
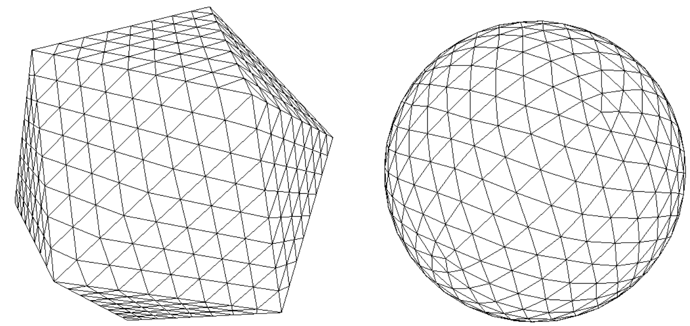


Results I

Simple tests

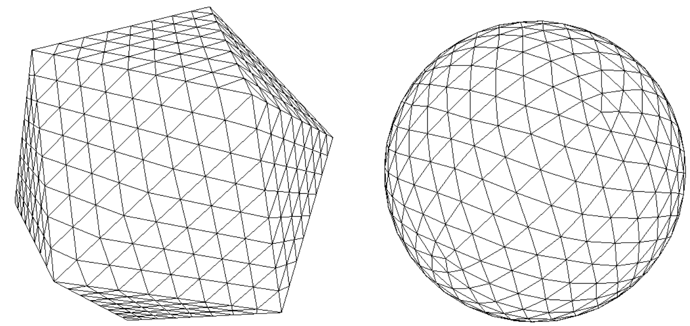
- sphere

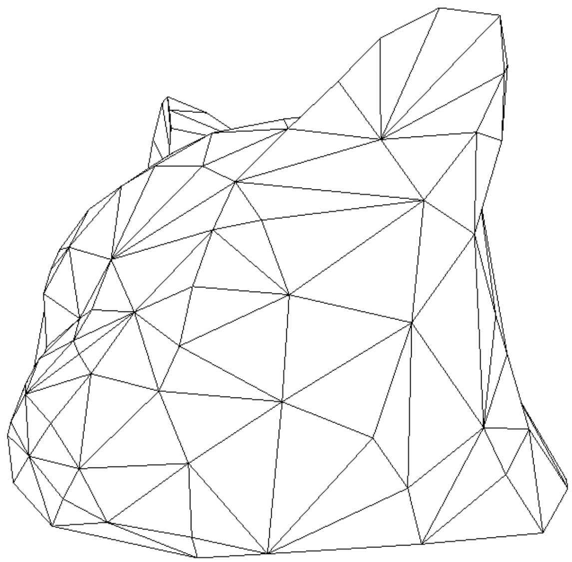


Results I

Simple tests

- sphere
- boundaries





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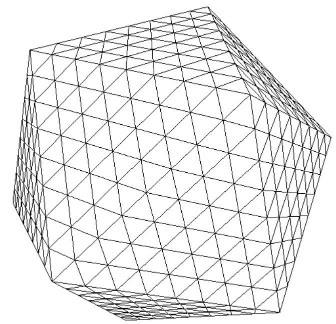

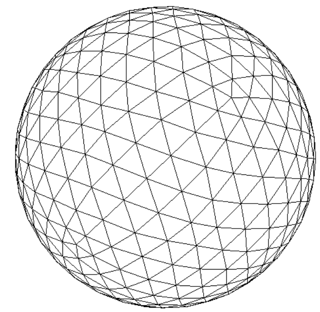

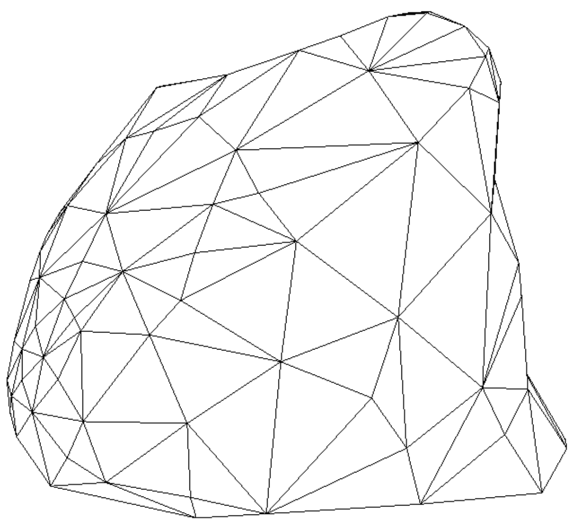

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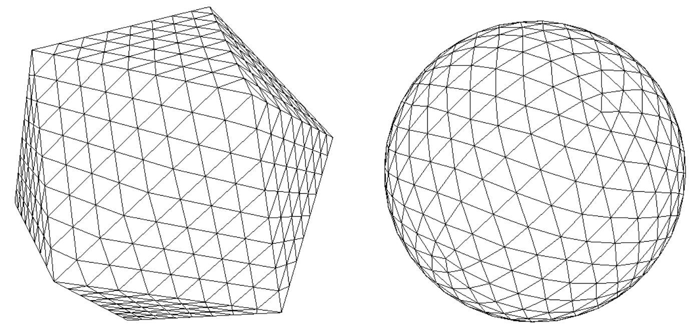

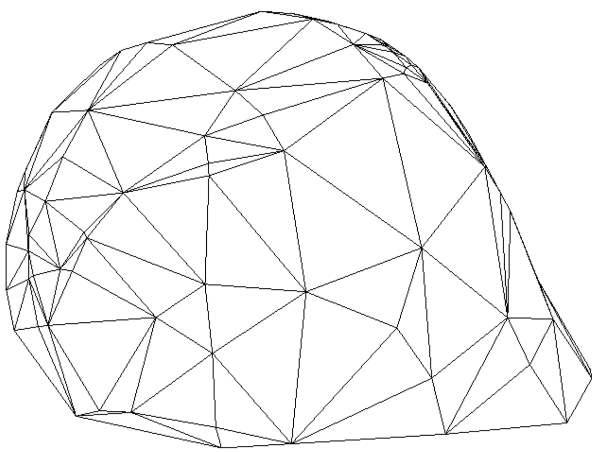

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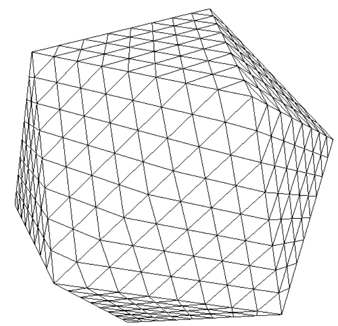

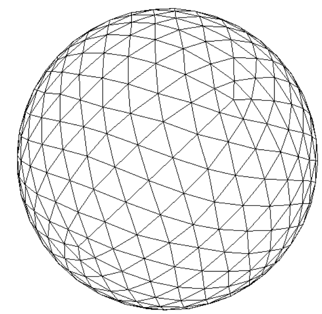

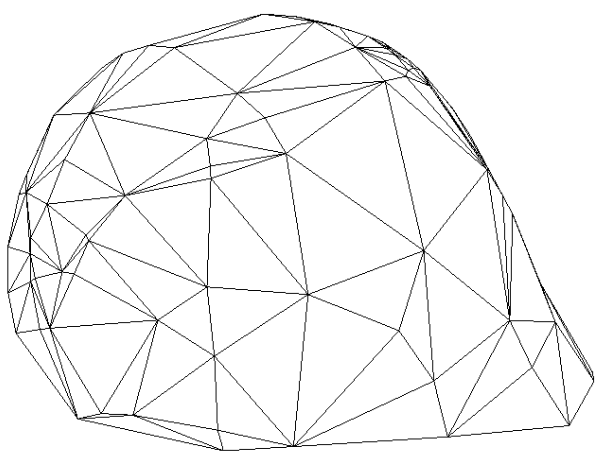

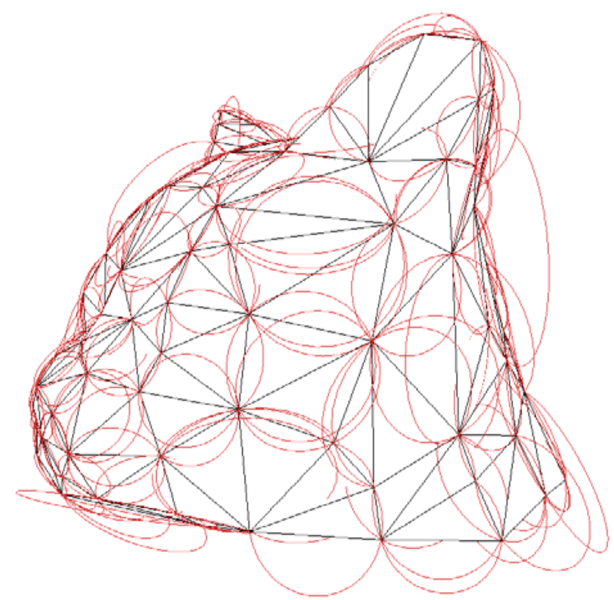

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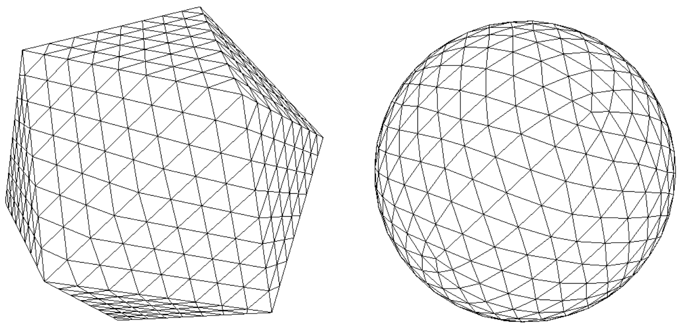

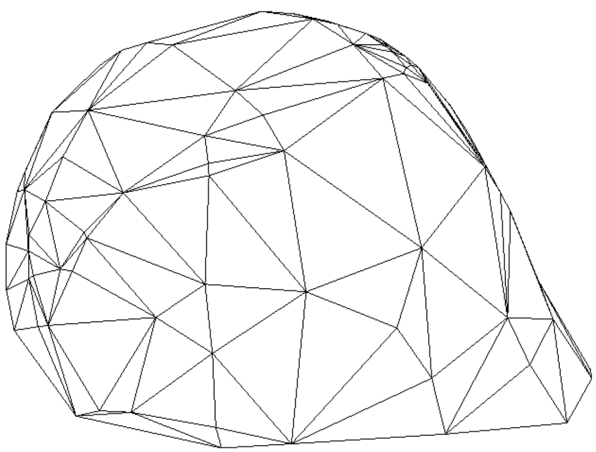

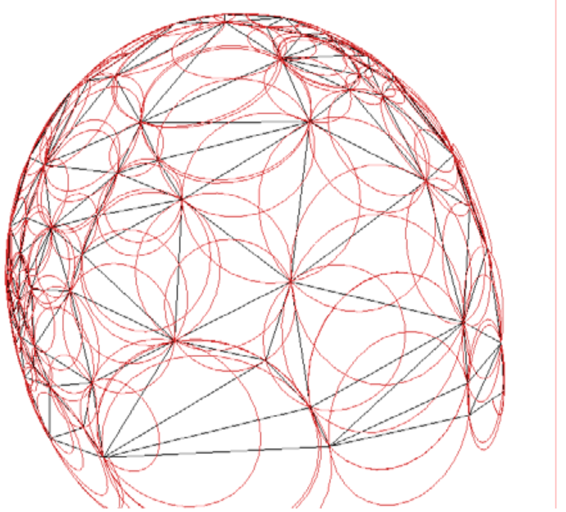

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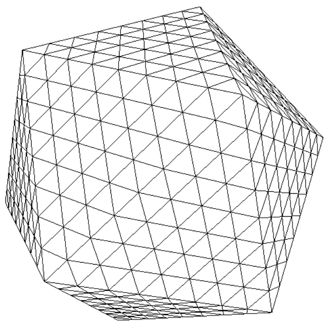
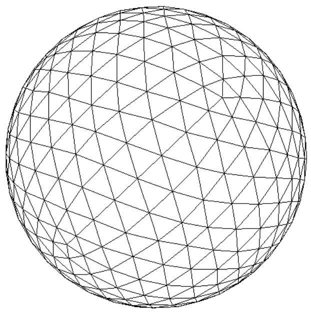
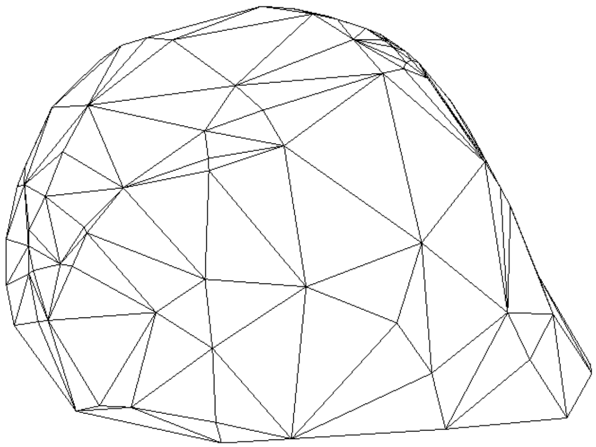
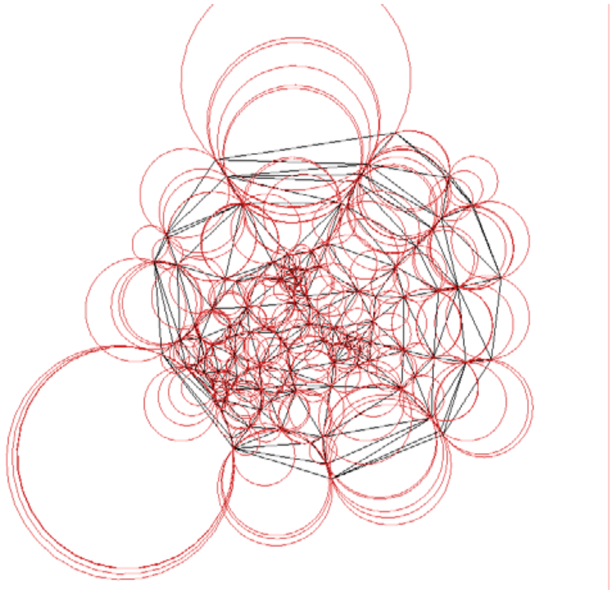


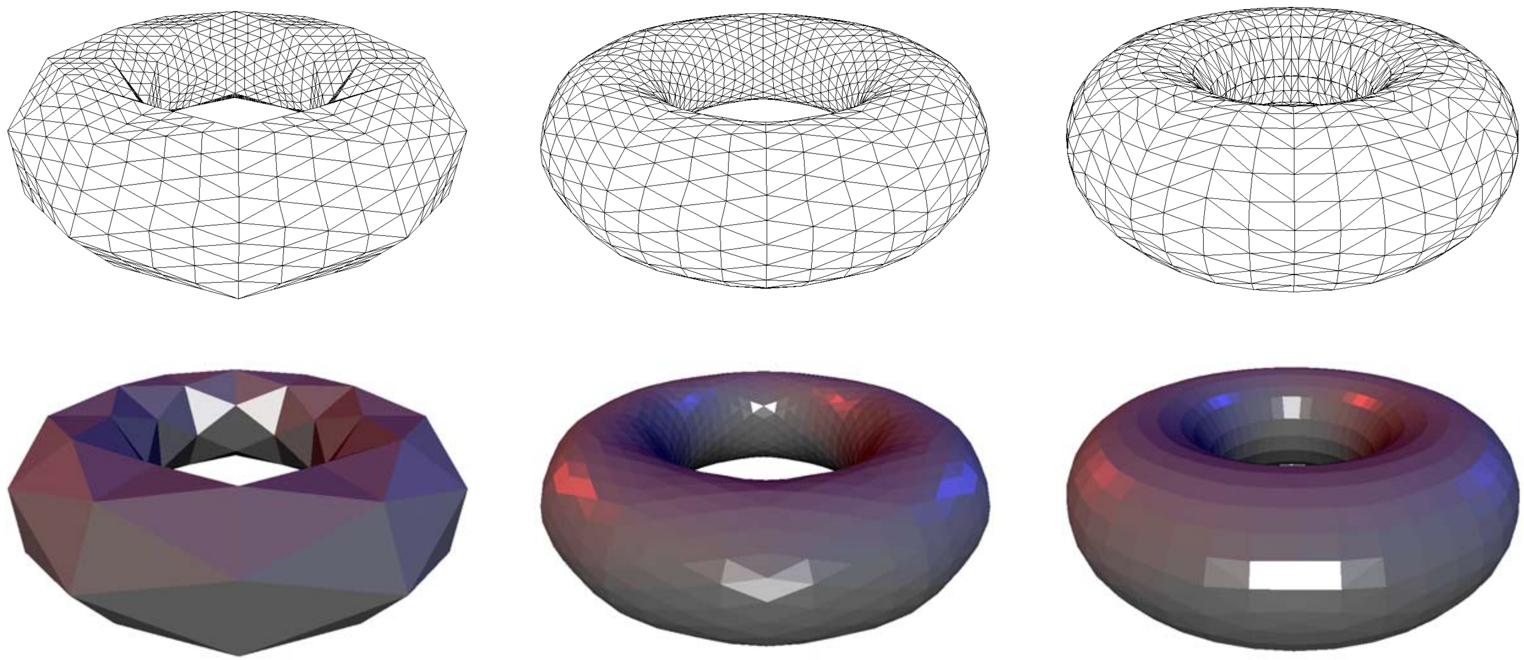






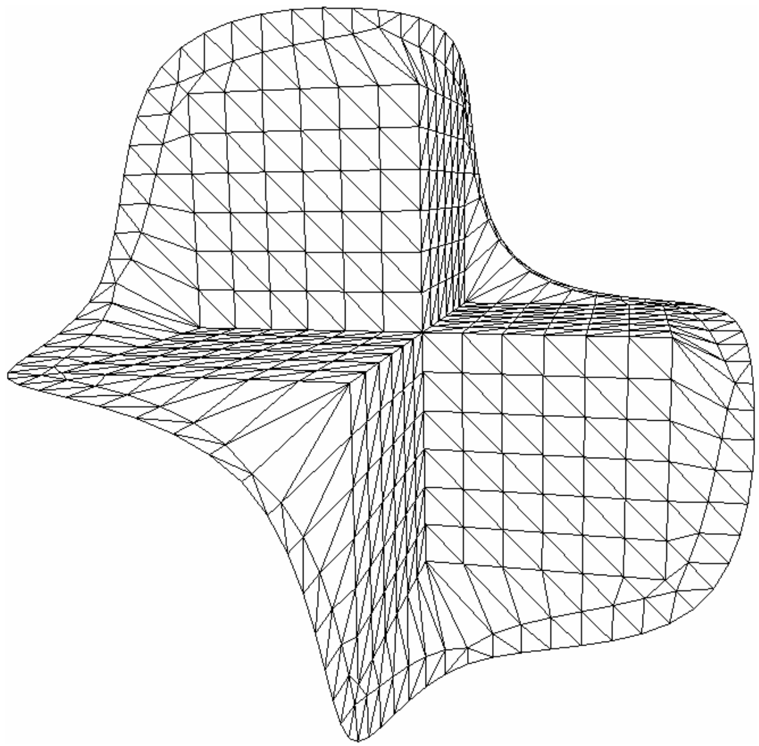
Results II

Curvature alignment



Results III

Hole filling



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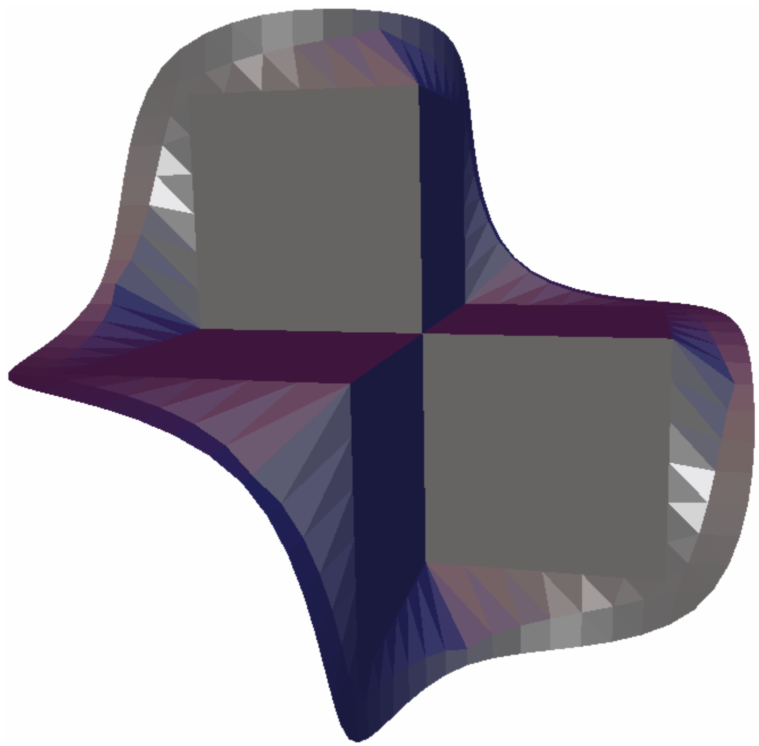

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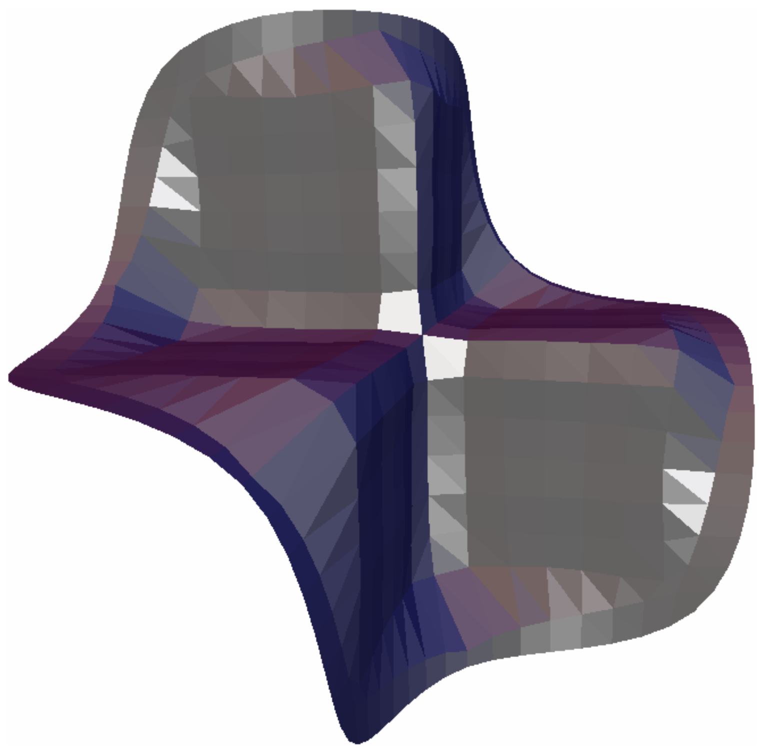

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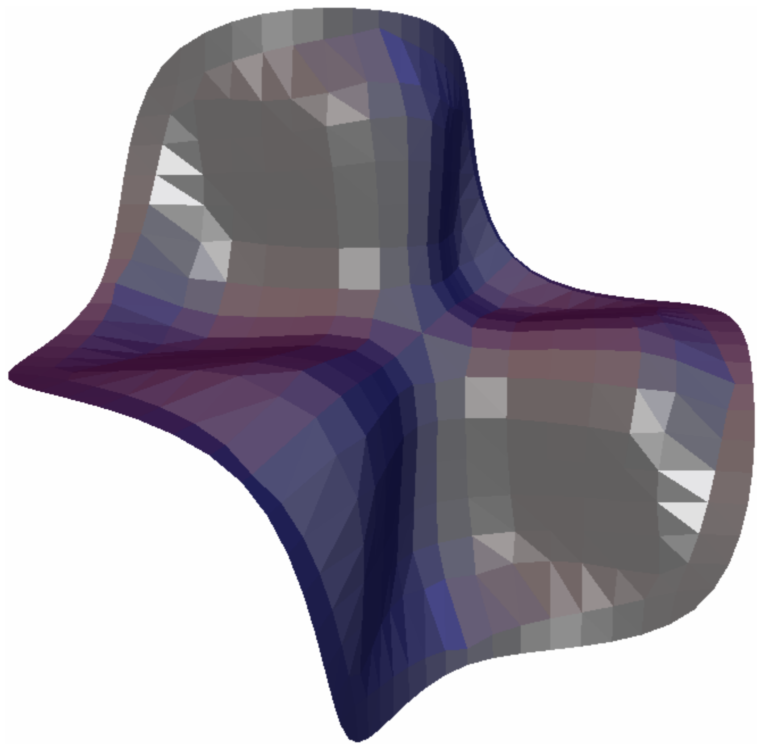

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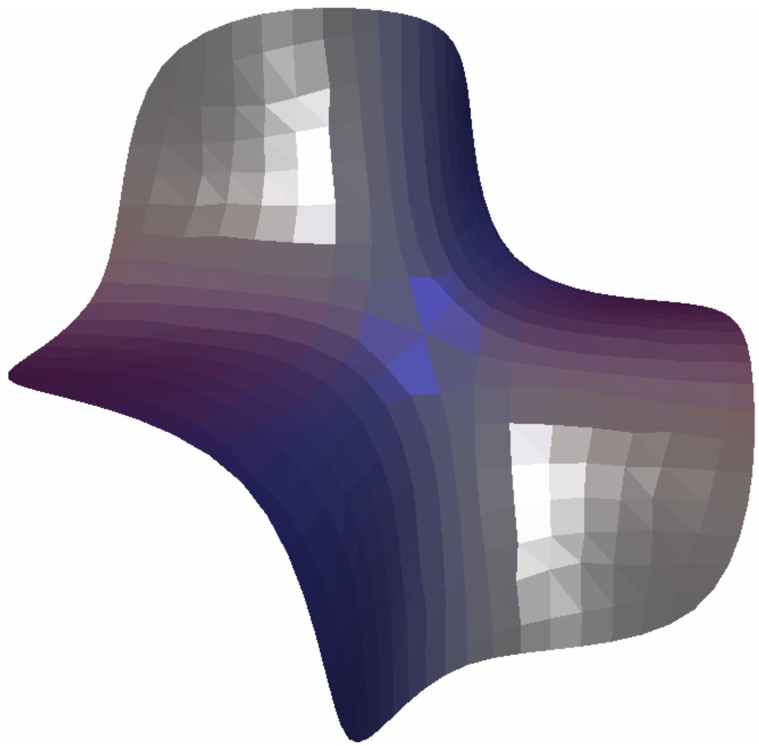

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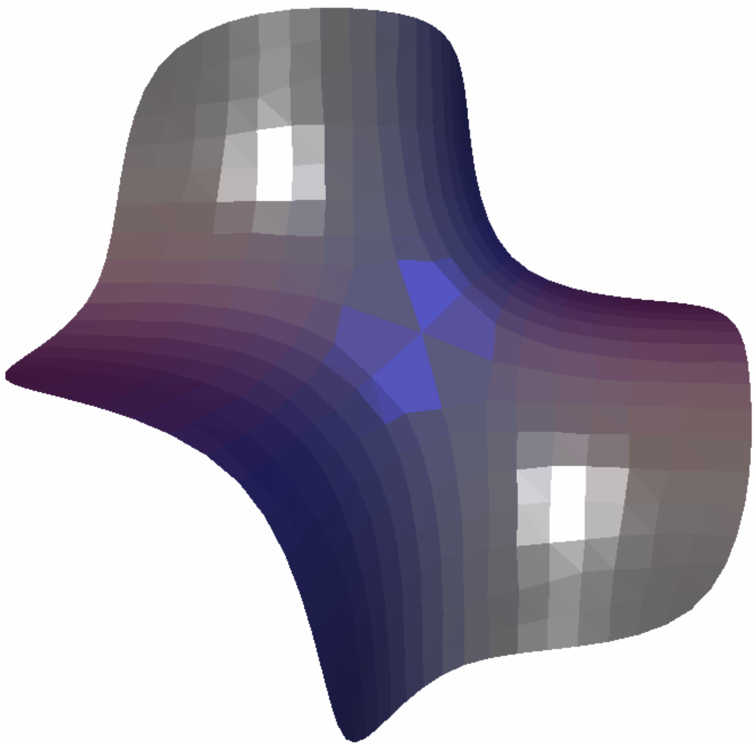

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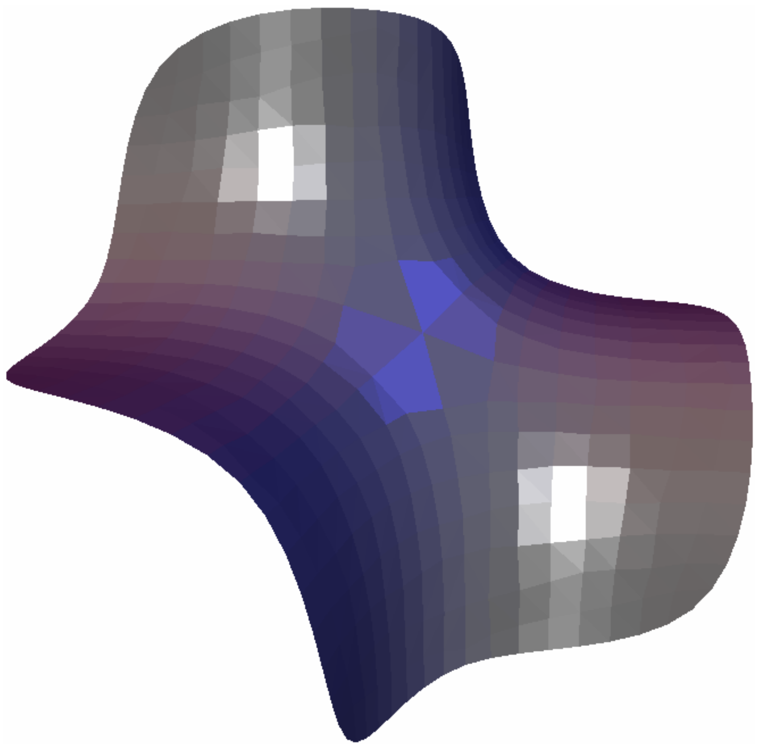
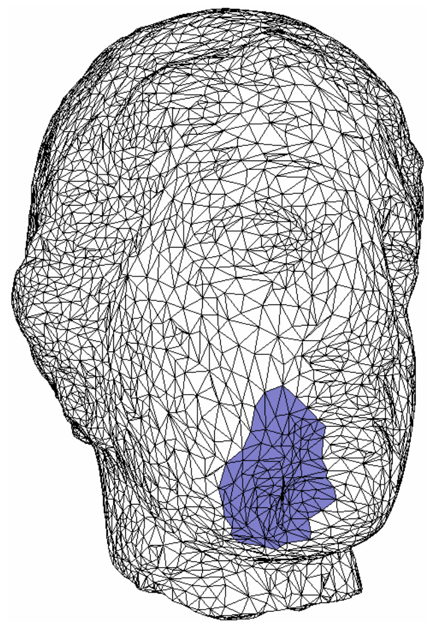


Inpainting



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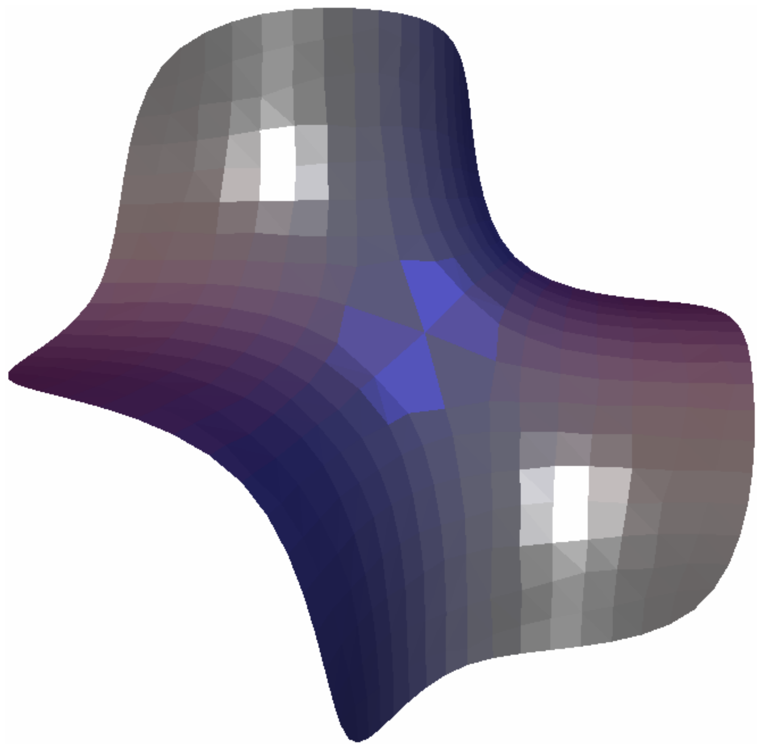

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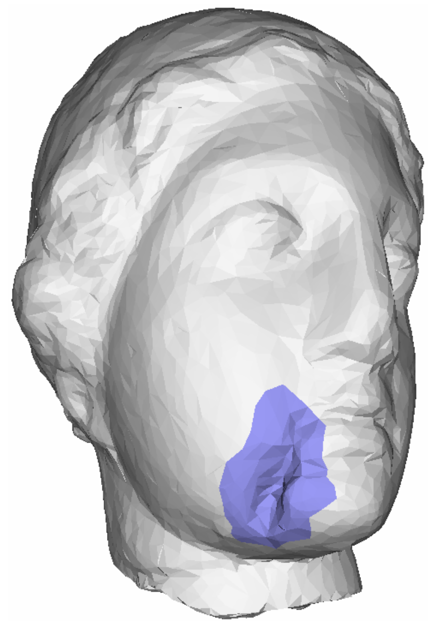

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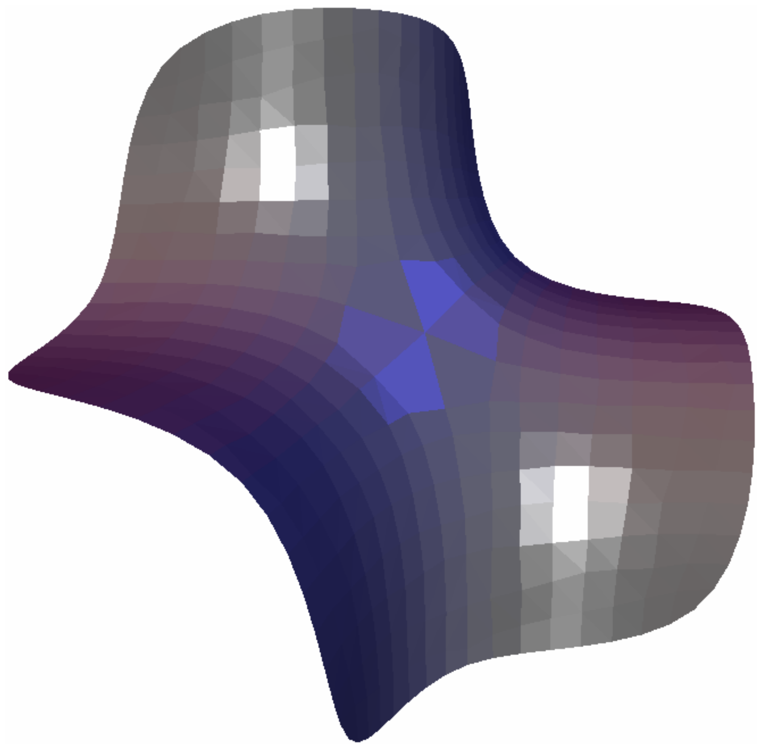

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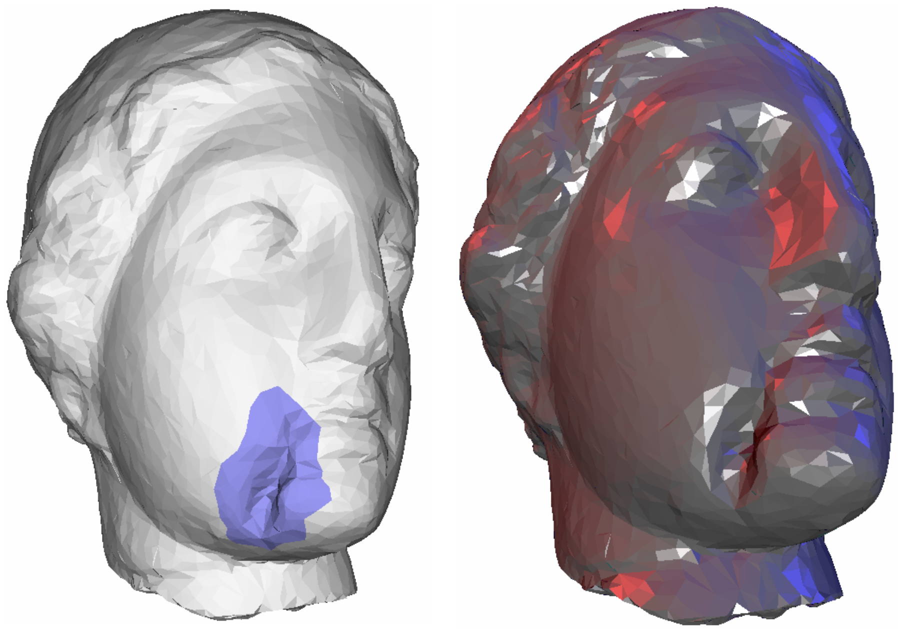

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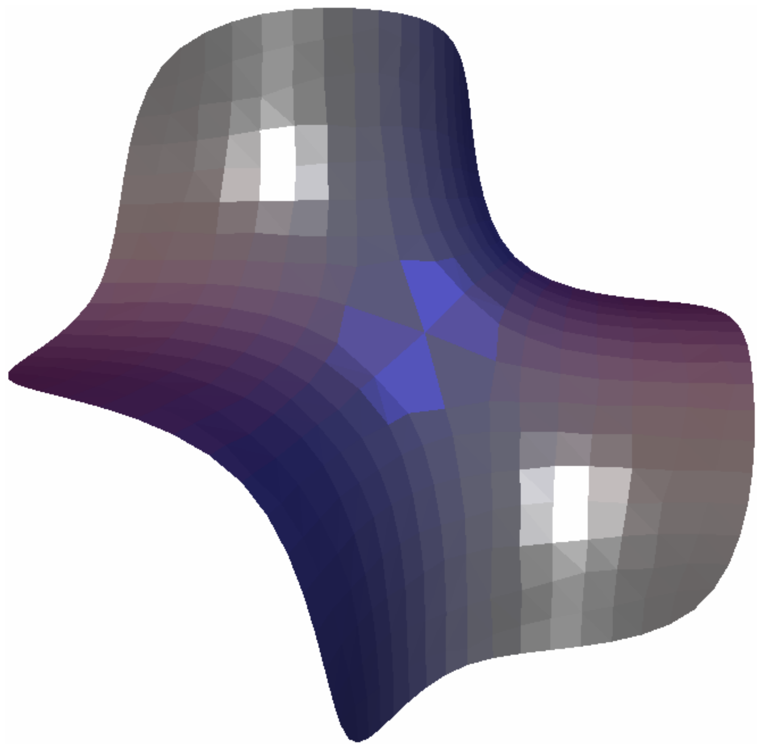

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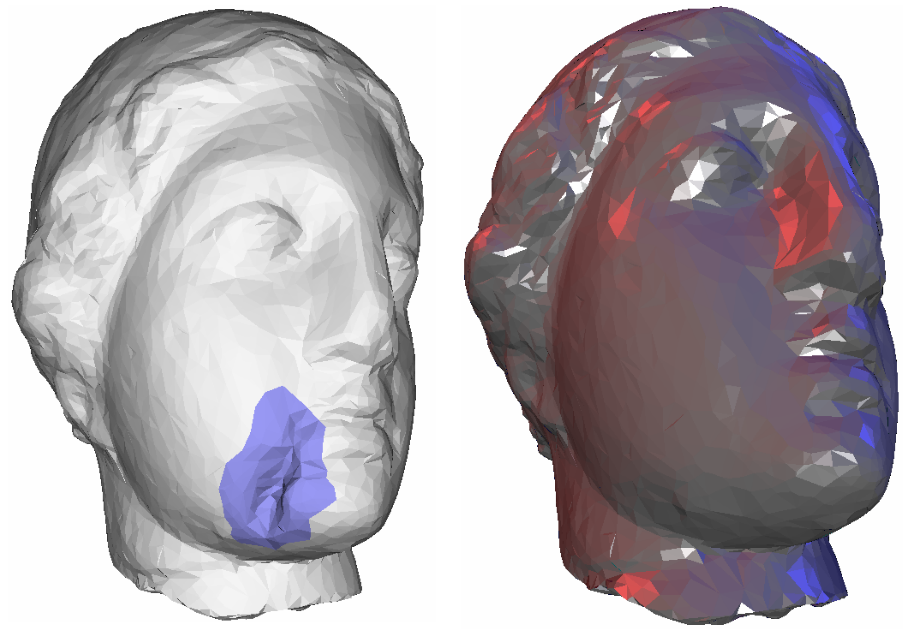
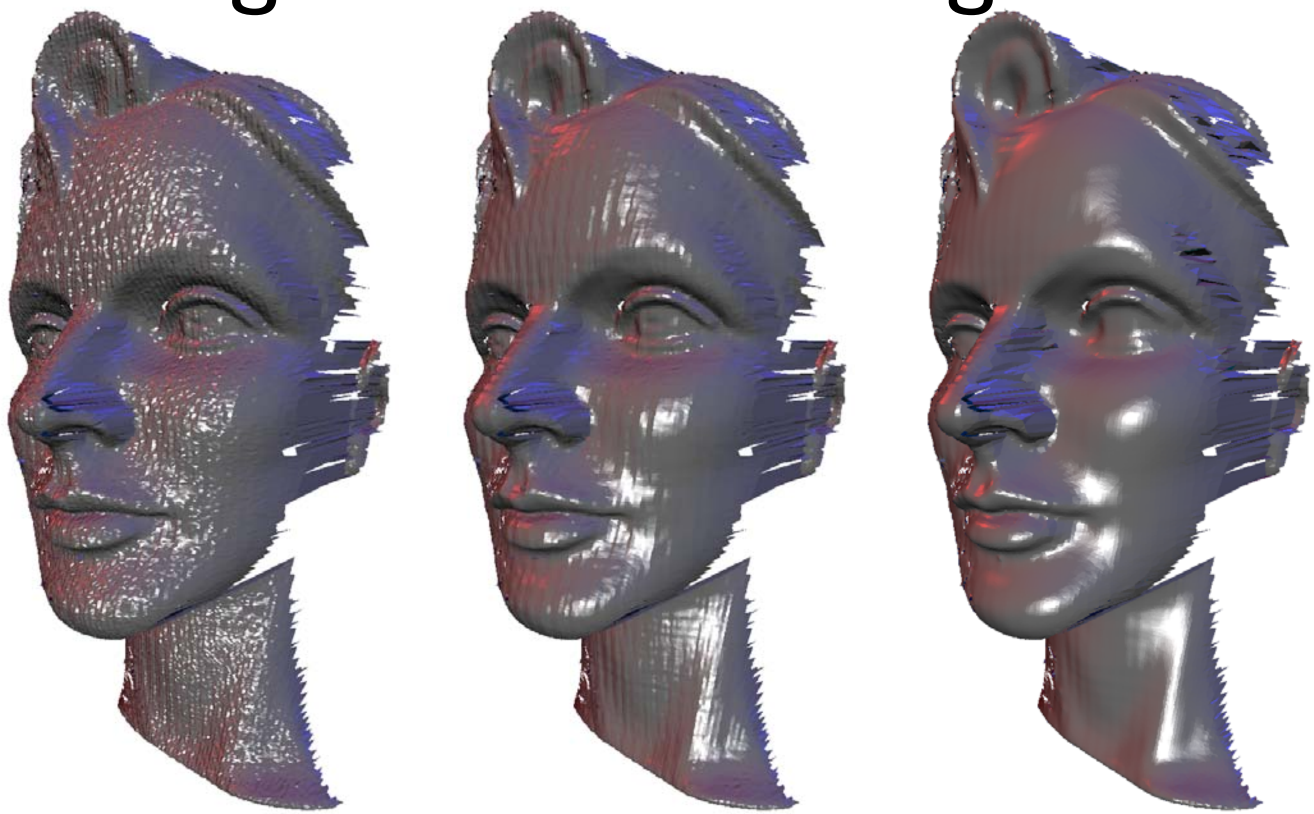


Results IV

Shrinkage free denoising



Summary

Discrete Willmore flow

- preserve symmetries: Möbius
- semi-implicit time stepping
- relevance in many geo. proc. areas
  - surface theory
  - variational geometric modeling
  - physical modeling

Outlook

Future work

- more boundary conditions
- incorporate reference curvature
- control of triangle quality
- variational subdivision
- numerical robustness

Circle Summary

Obey the geometry

- what geometry do the objects of interest belong to?
  - conformal parameterization
  - curvature energies
- circles and the angles they make with one another
- complete non-linear treatment