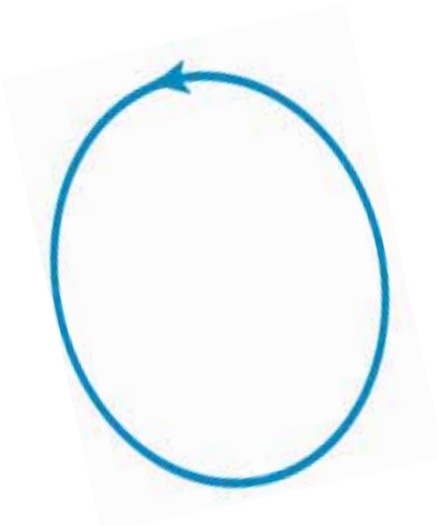
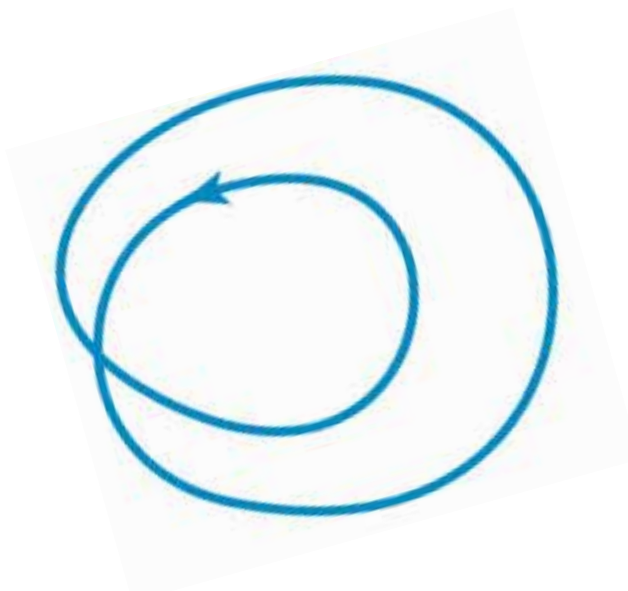
A First Look at DDG: Discrete Curves

Eitan Grinspun, Columbia University

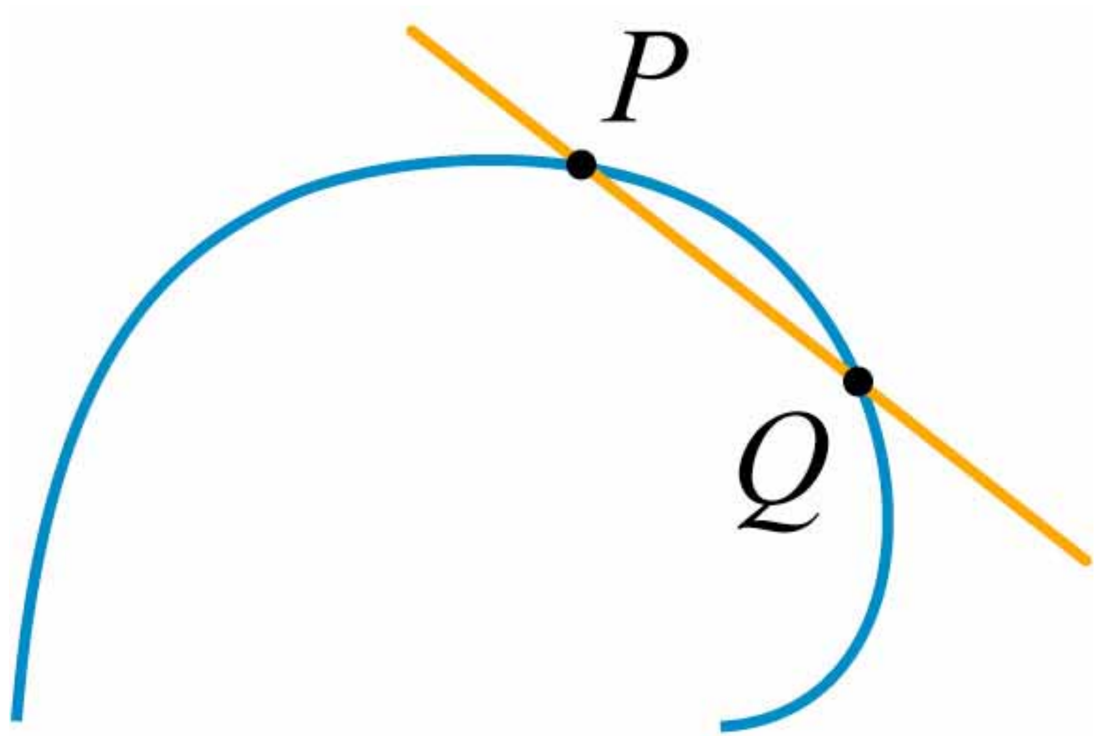
Part I: plane curves





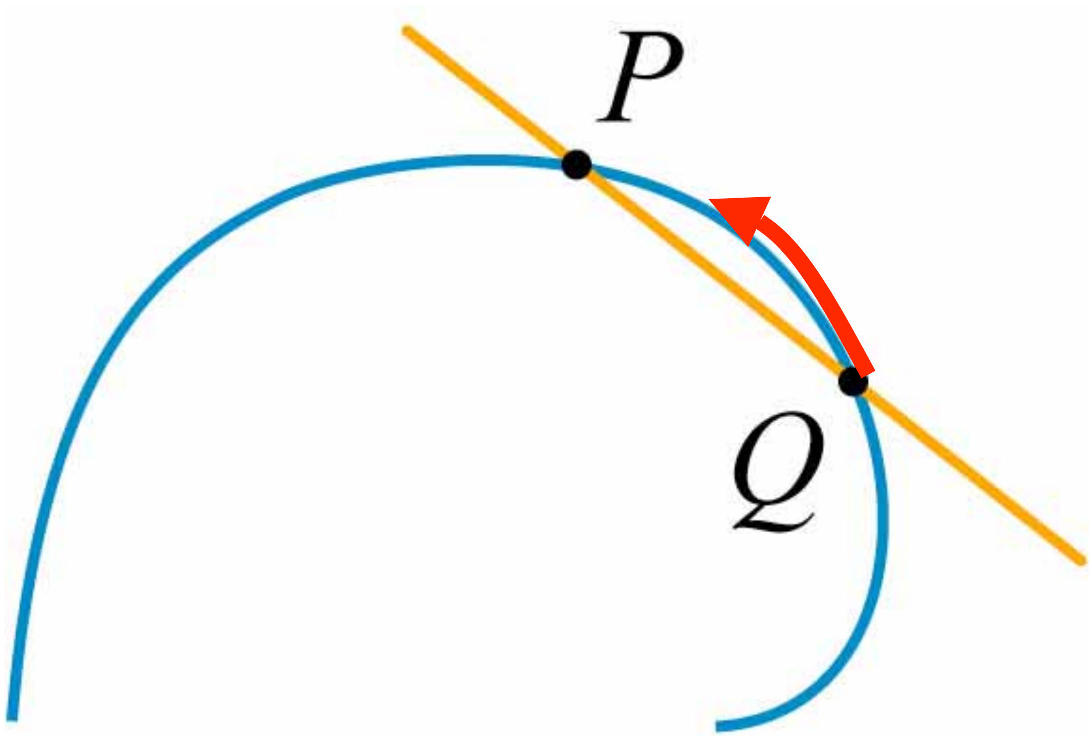
Secant

A line through two points on the curve.



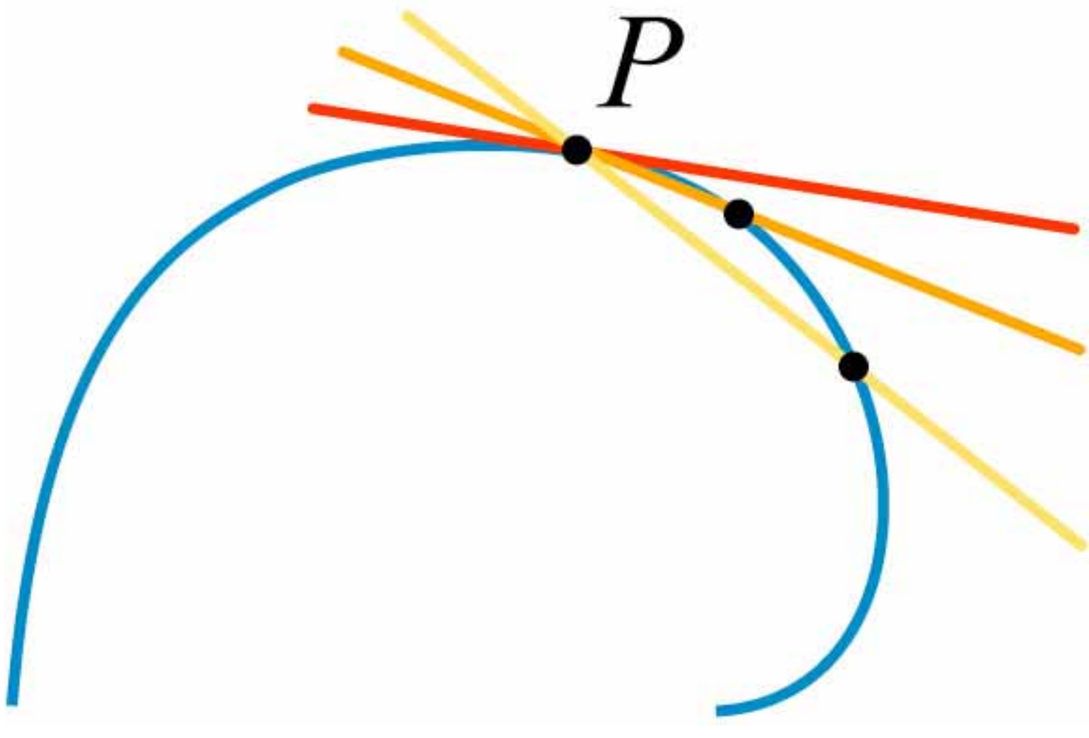
Secant

A line through two points on the curve.



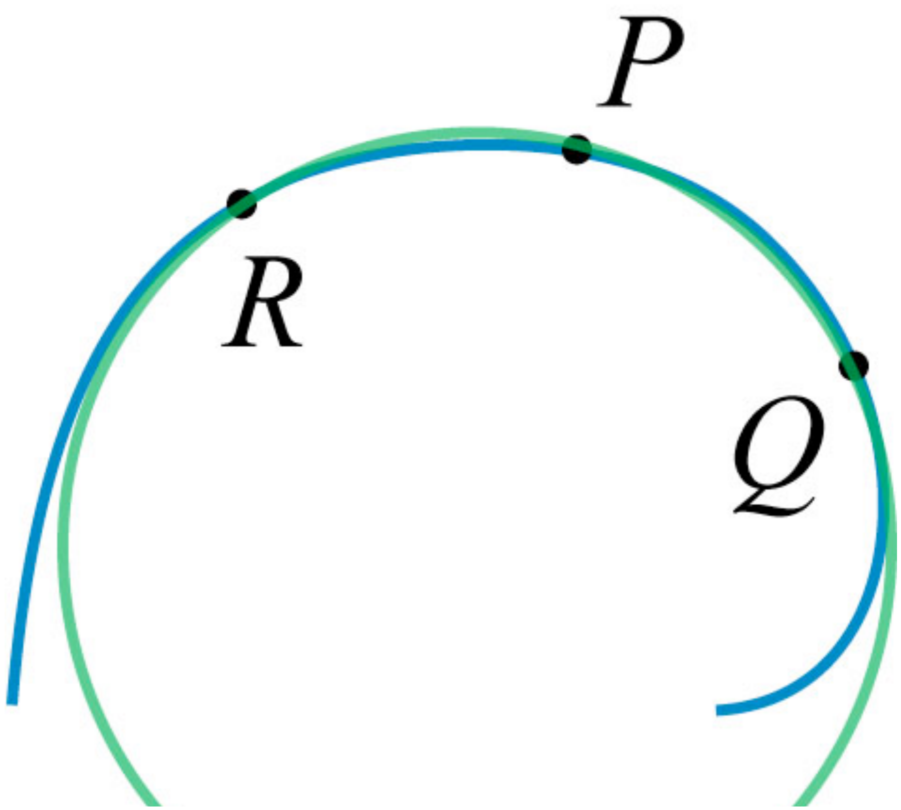
Tangent, the first approximant

The limiting secant as the two points come together.



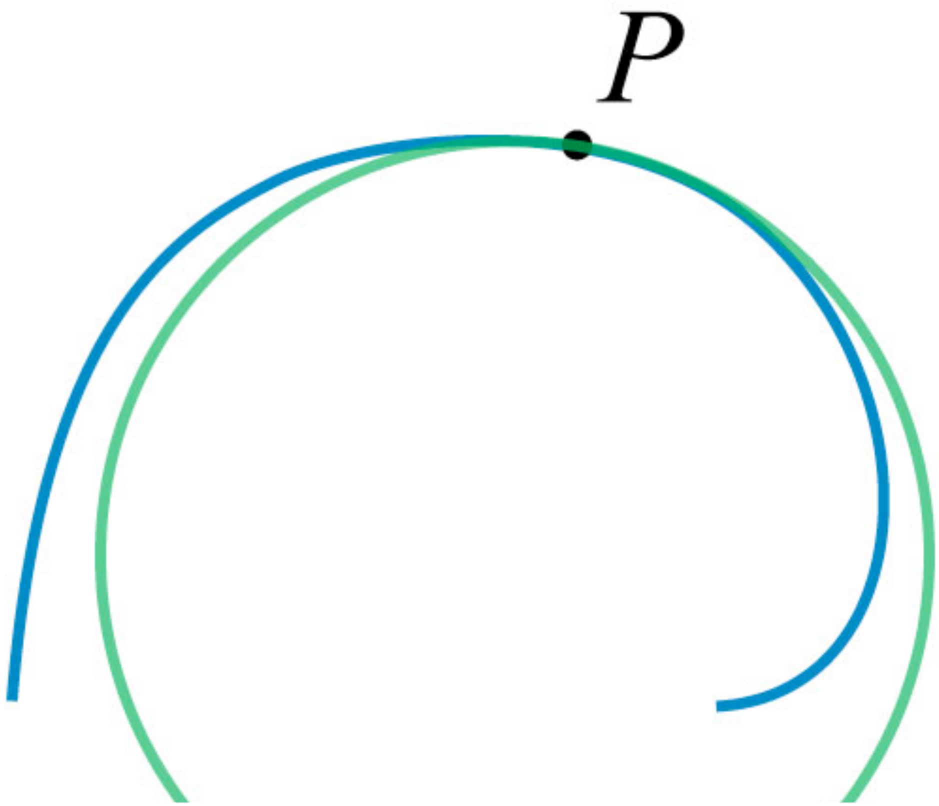
Circle of curvature

Consider the circle passing through three points on the curve…

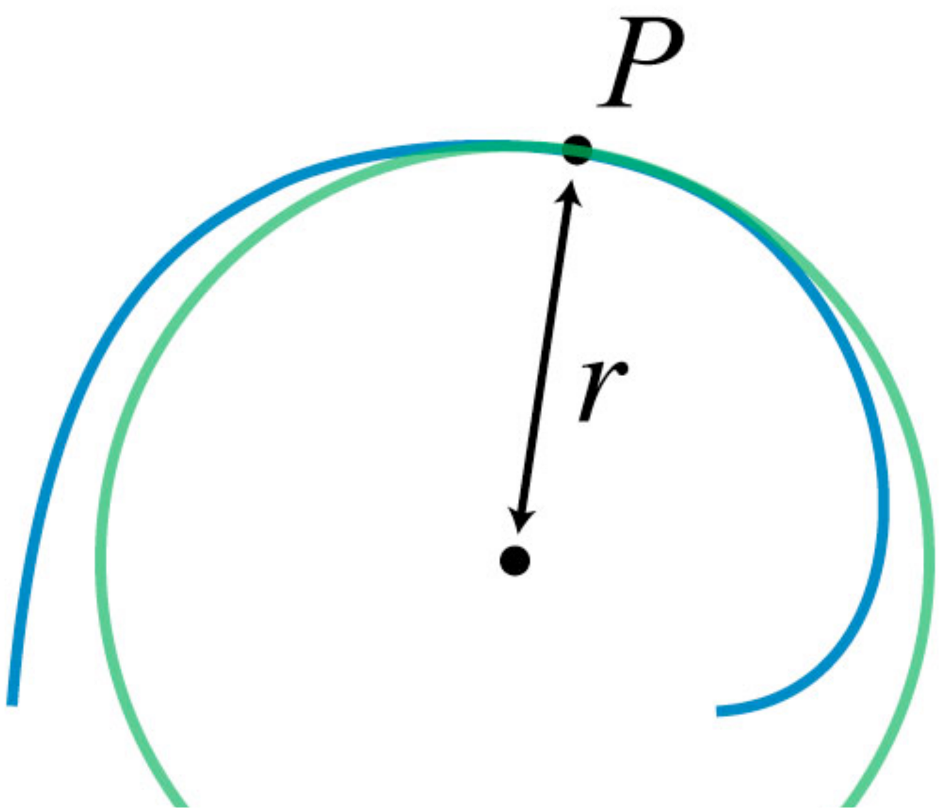


Circle of curvature

…the limiting circle as three points come together.



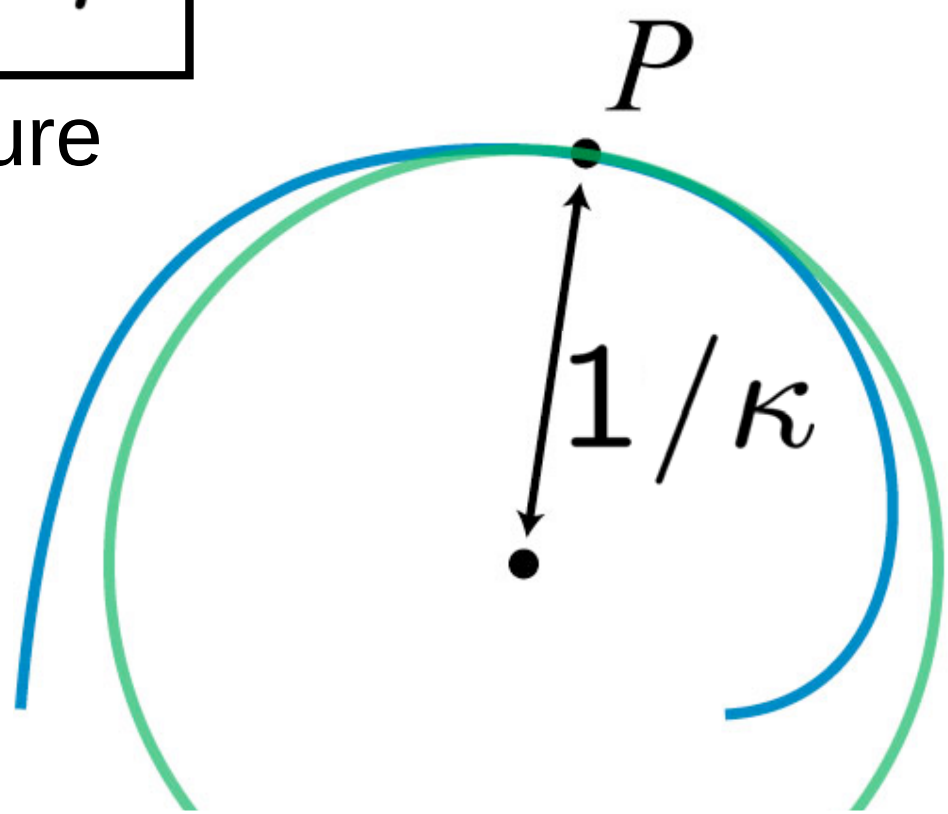
Radius of curvature, r



Radius of curvature, $r = 1/\kappa$

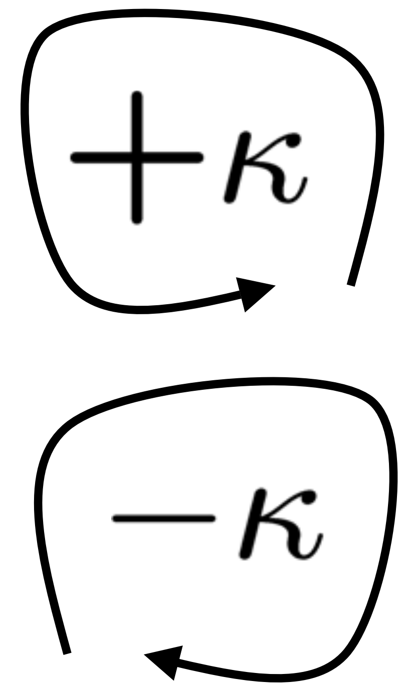
$$\kappa = \frac{1}{r}$$

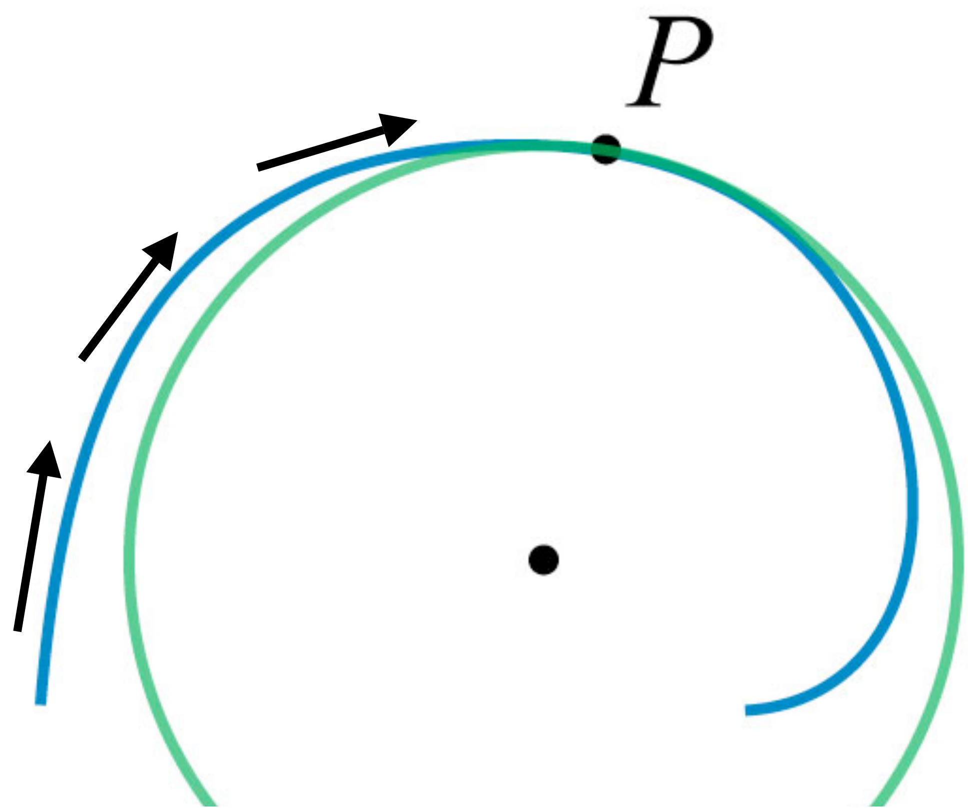
Curvature



Signed curvature

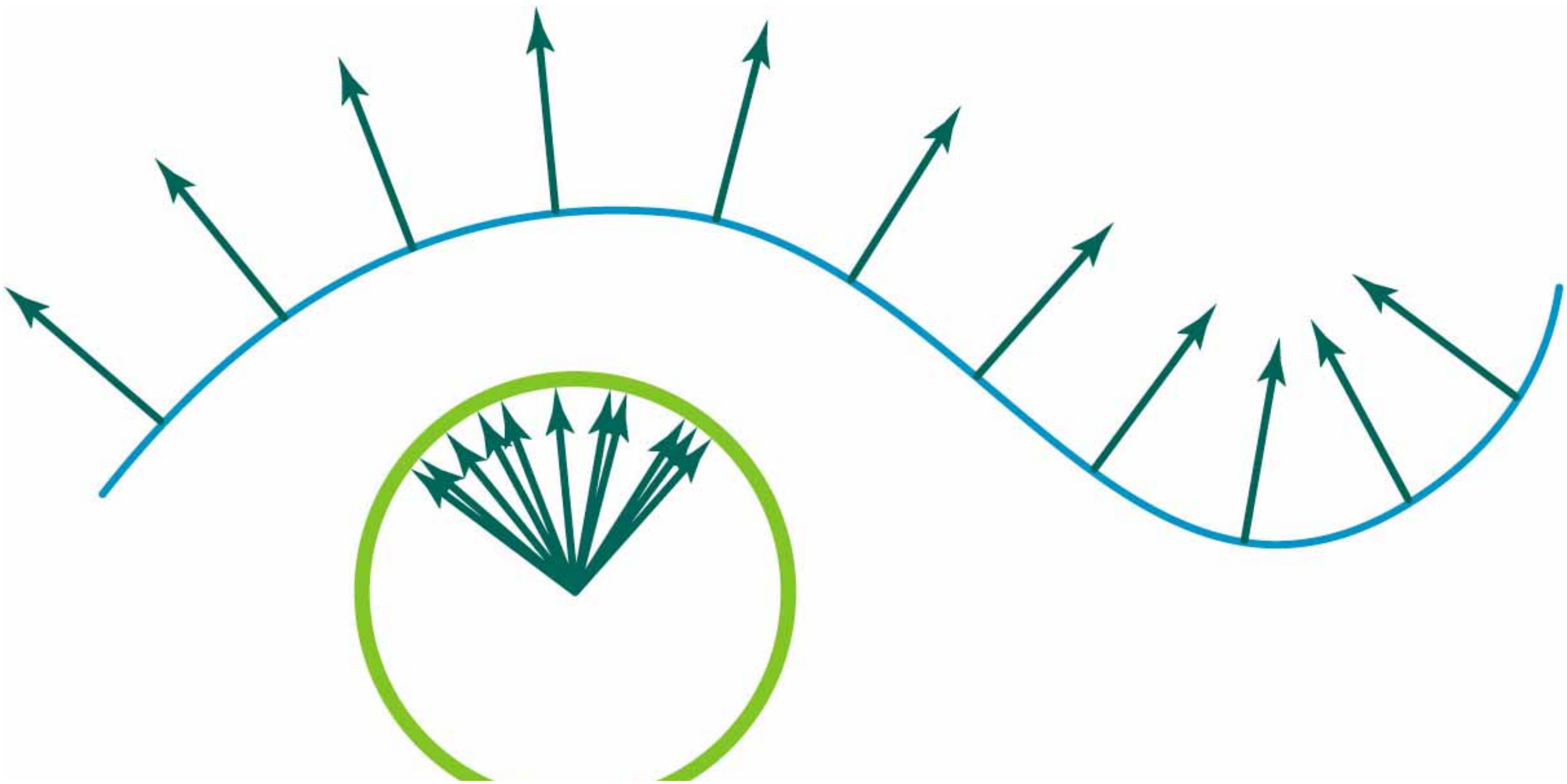
Sense of traversal along curve.





Gauß map, $\hat{\mathbf{n}}(\mathbf{x})$

Point on curve maps to point on unit circle.

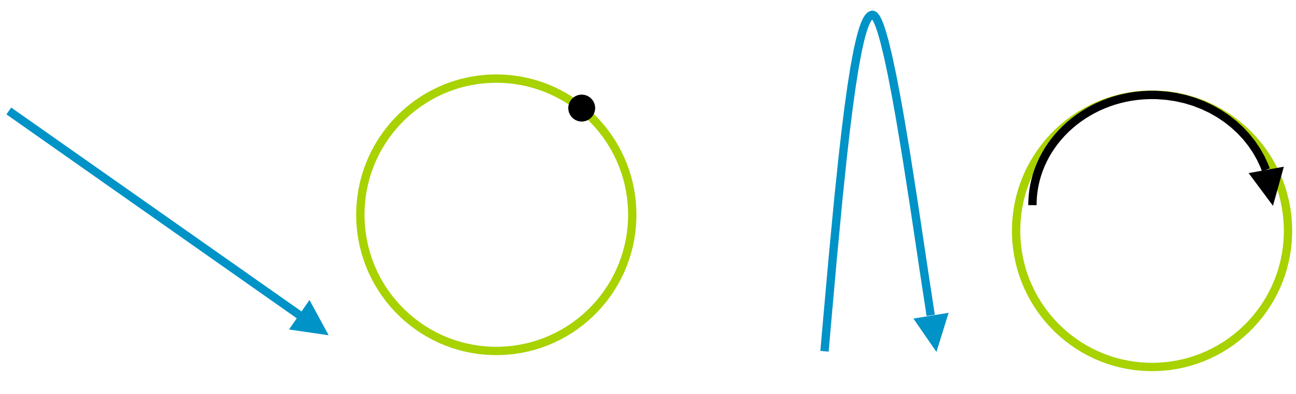


Shape operator

Change in normal as we slide along curve

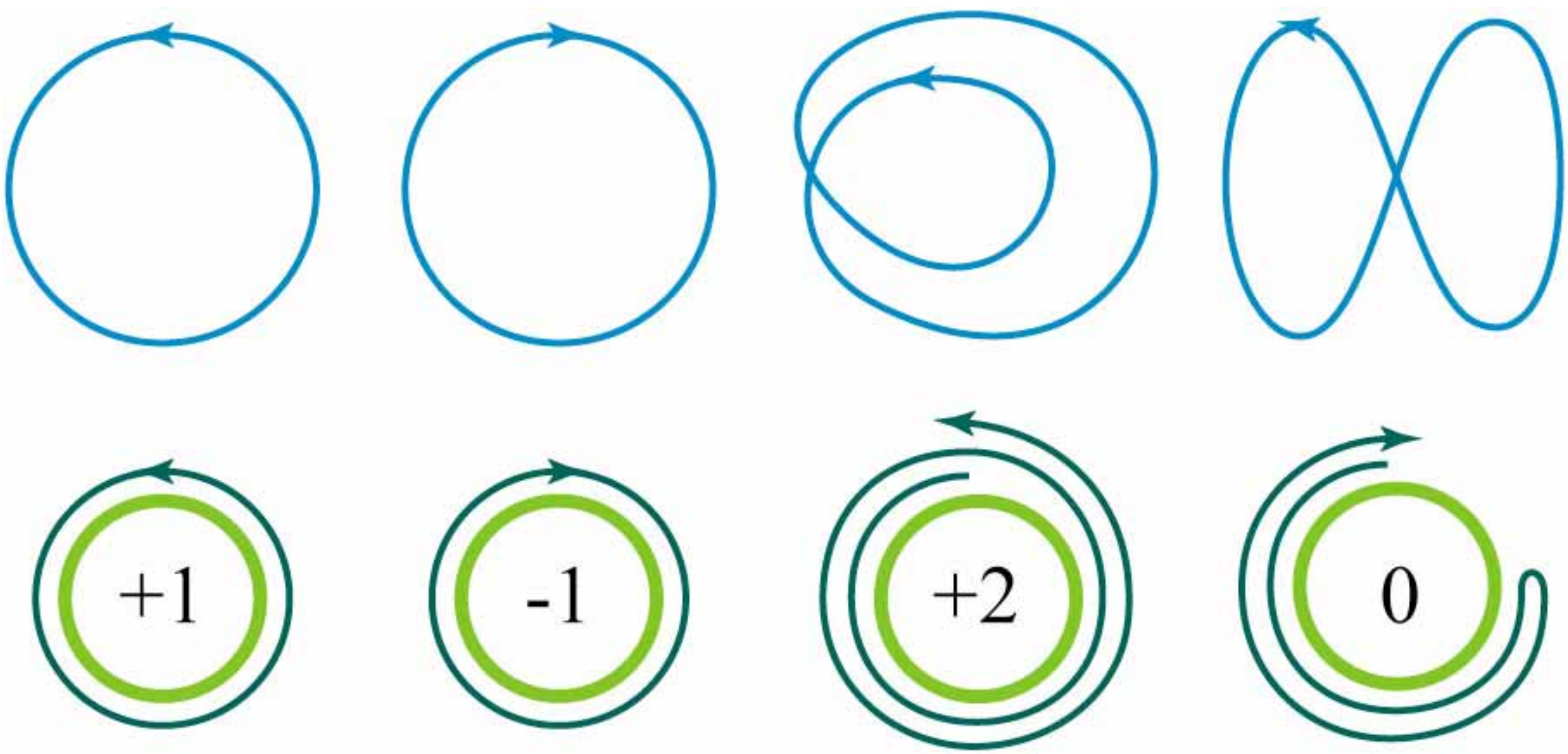
$$\mathbf{S}(\mathbf{v}) = -D_{\mathbf{v}}\hat{\mathbf{n}}$$

- describes curvature



Turning number, k

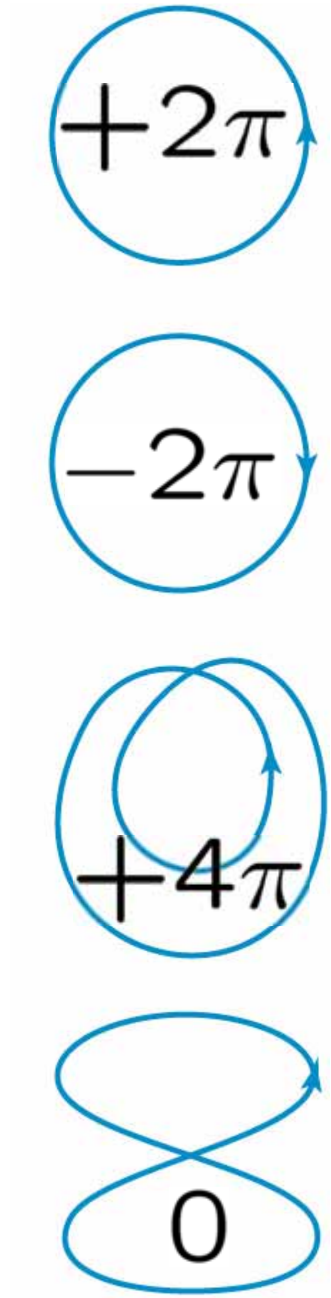
Number of orbits in Gaussian image.



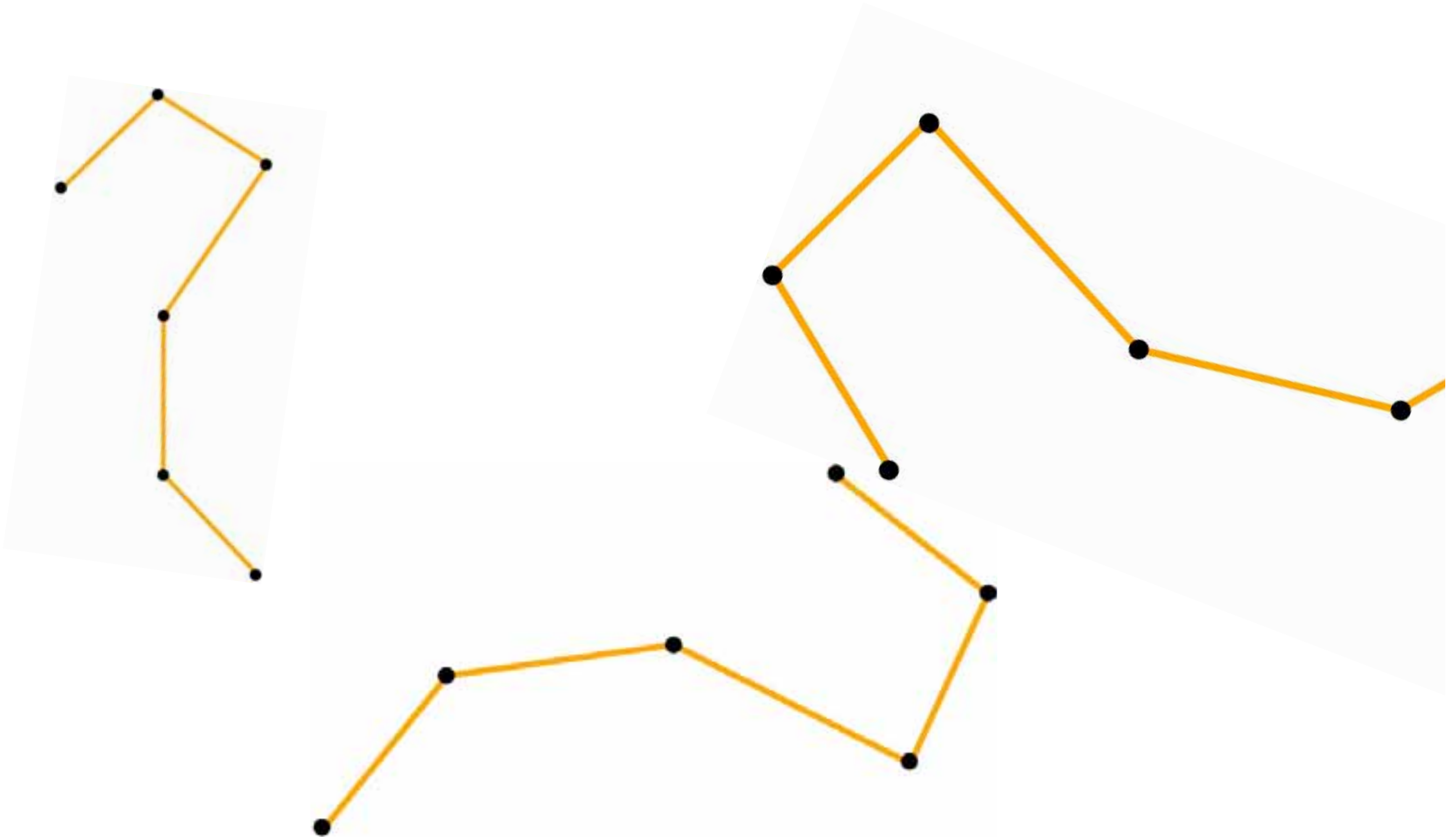
Turning number theorem

$$\int_{\Omega} \kappa ds = 2\pi k$$

For a closed curve,
the integral of curvature is
an integer multiple of 2π.

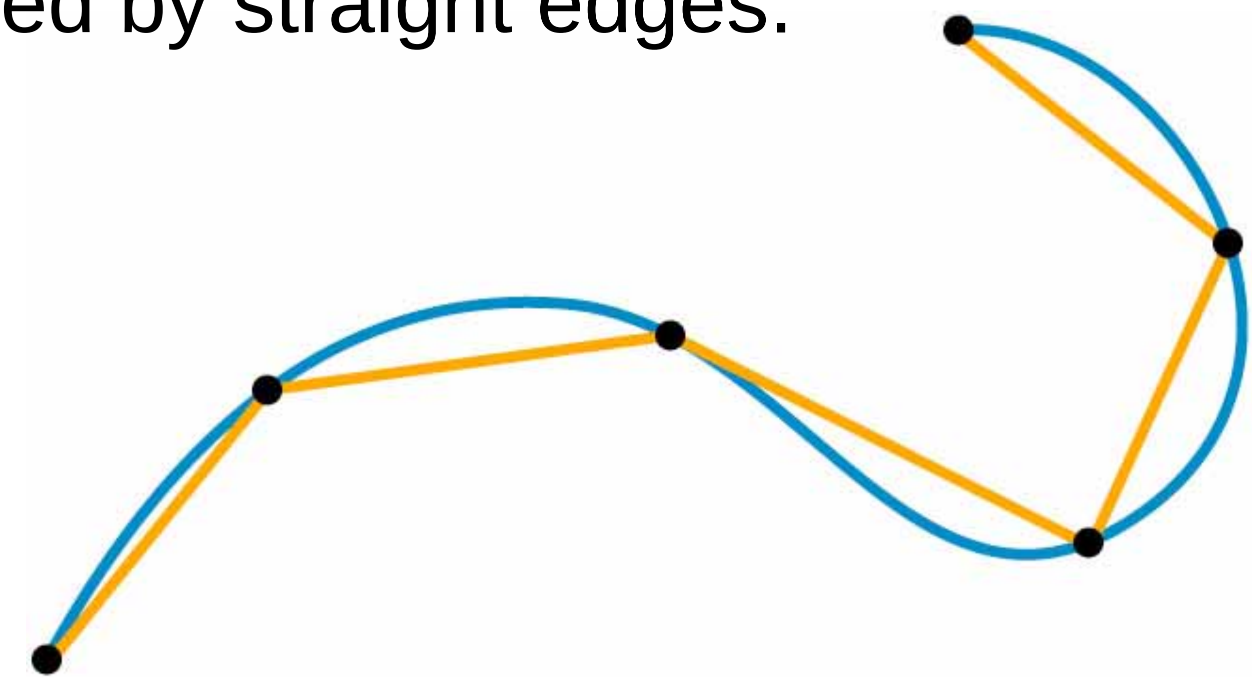


Part II: discrete plane curves



Inscribed polygon, p

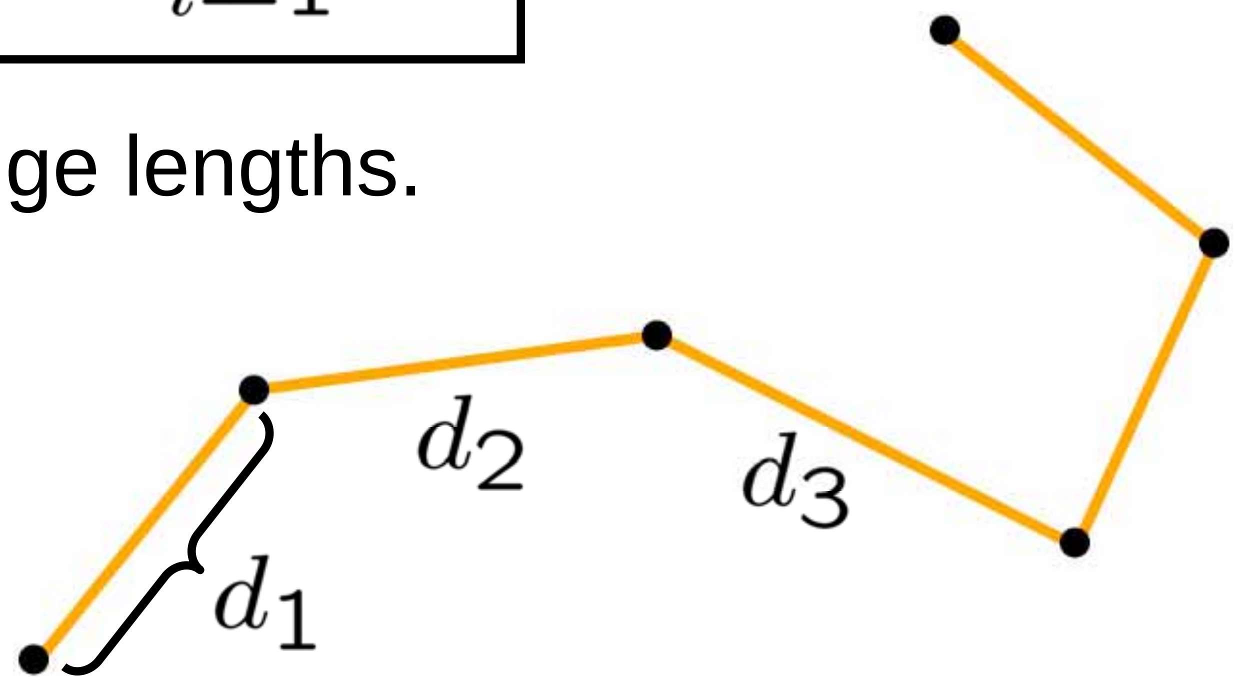
Finite number of vertices
each lying on the curve,
connected by straight edges.



The length of a discrete curve

$$len(p) = \sum_{i=1}^{n} d_i$$

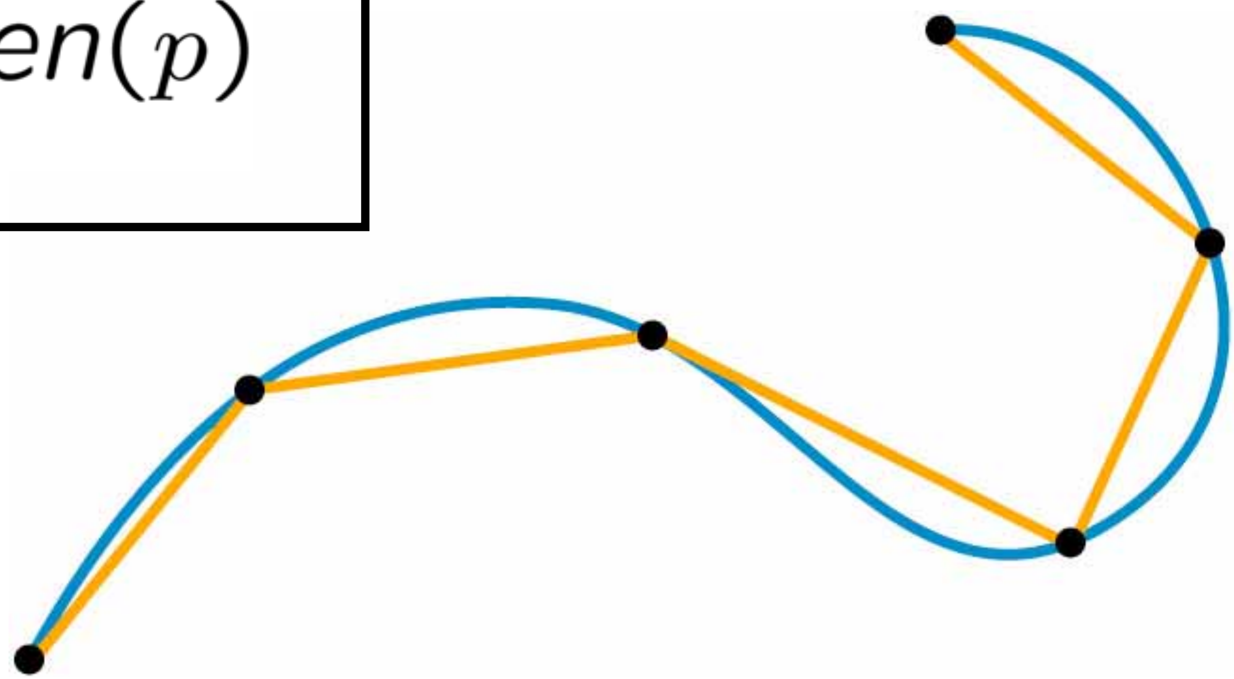
Sum of edge lengths.



The length of a continuous curve

Length of longest of all inscribed polygons.

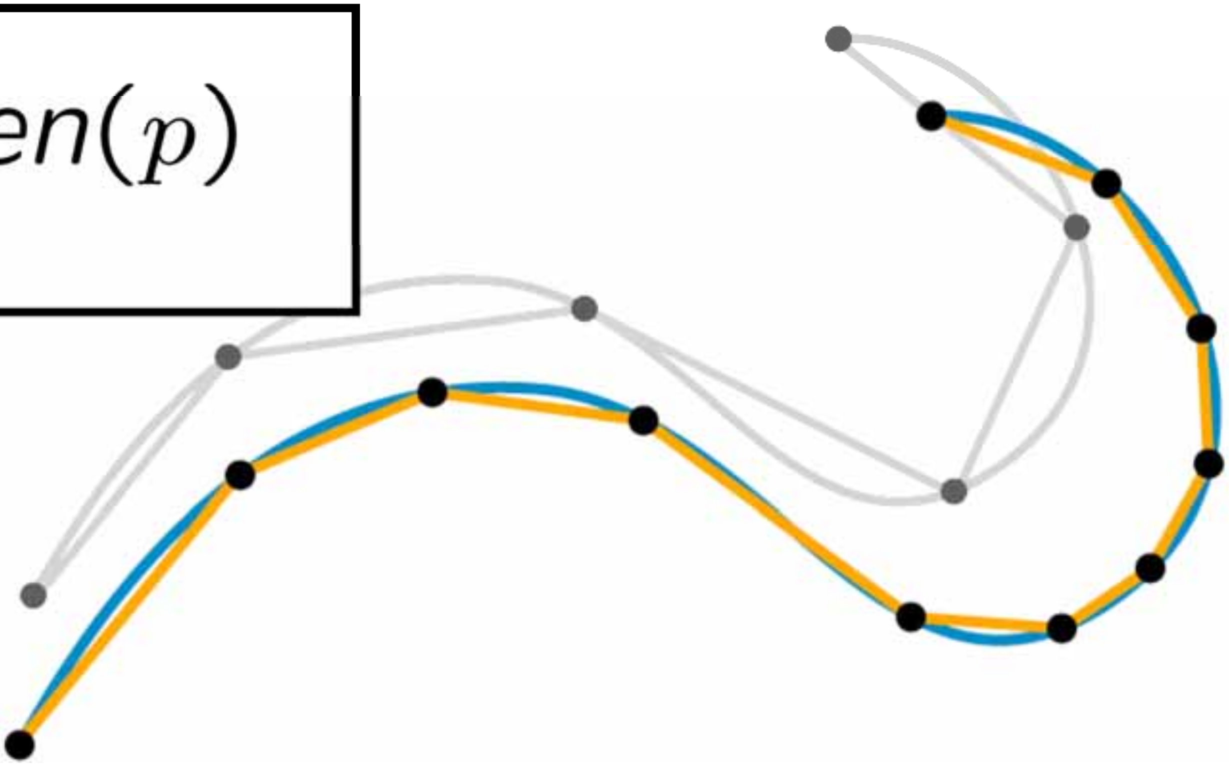
$$\sup_{p} len(p)$$



The length of a continuous curve

…or take limit over refinement sequence.

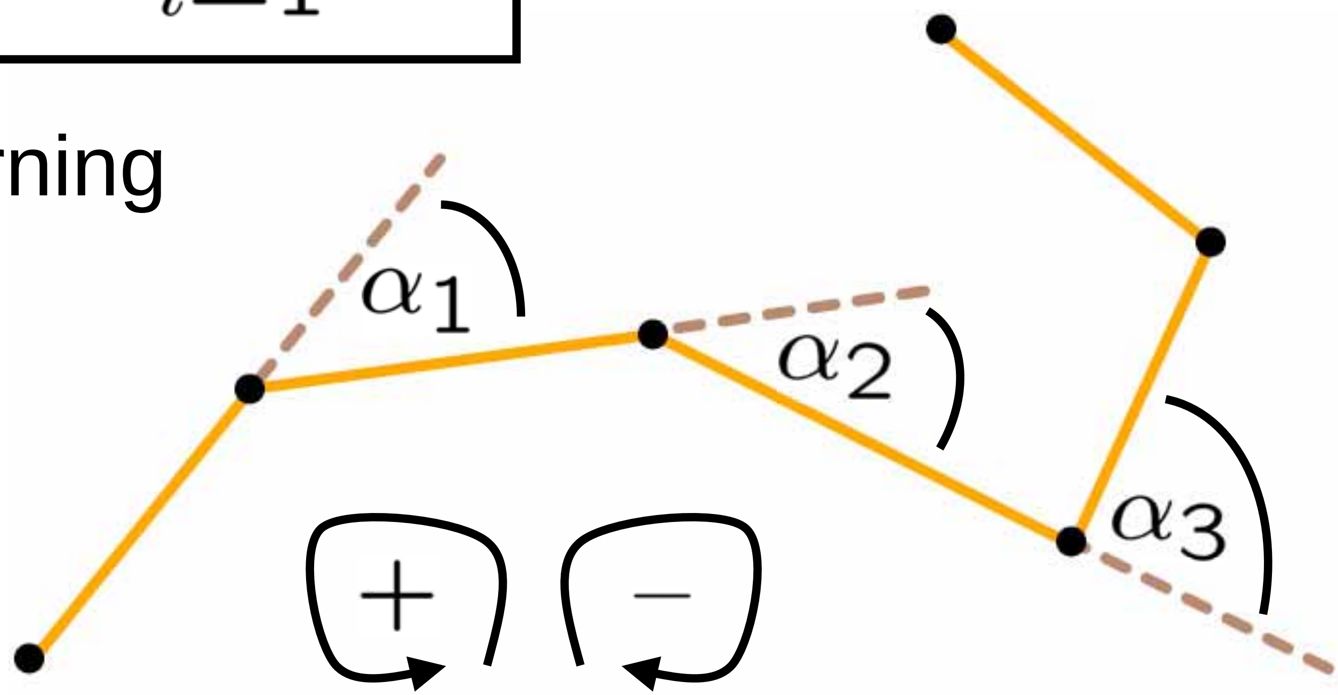
$$\lim_{h \to 0} len(p)$$



Total signed curvature

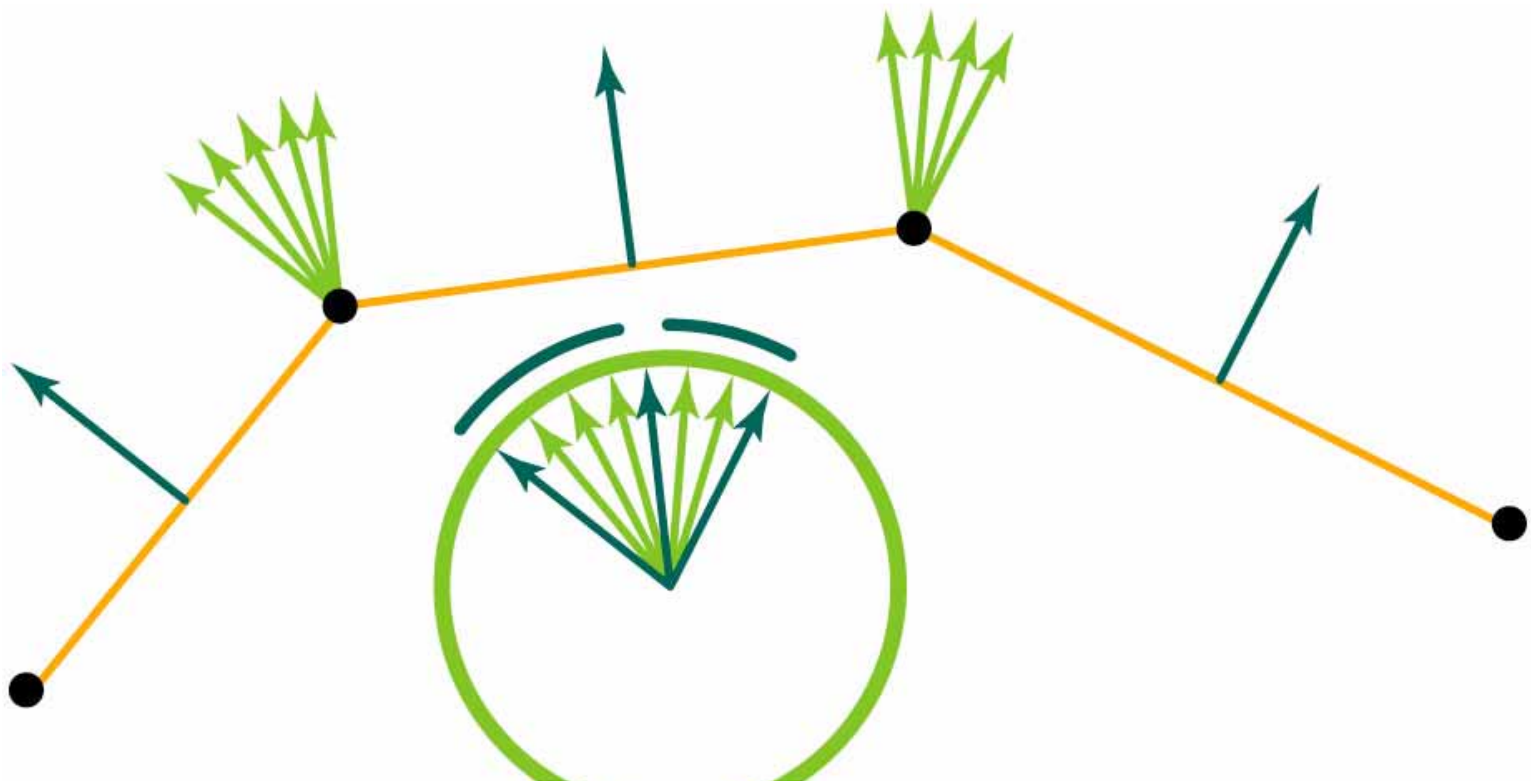
$$tsc(p) = \sum_{i=1}^{n} \alpha_i$$

Sum of turning angles.



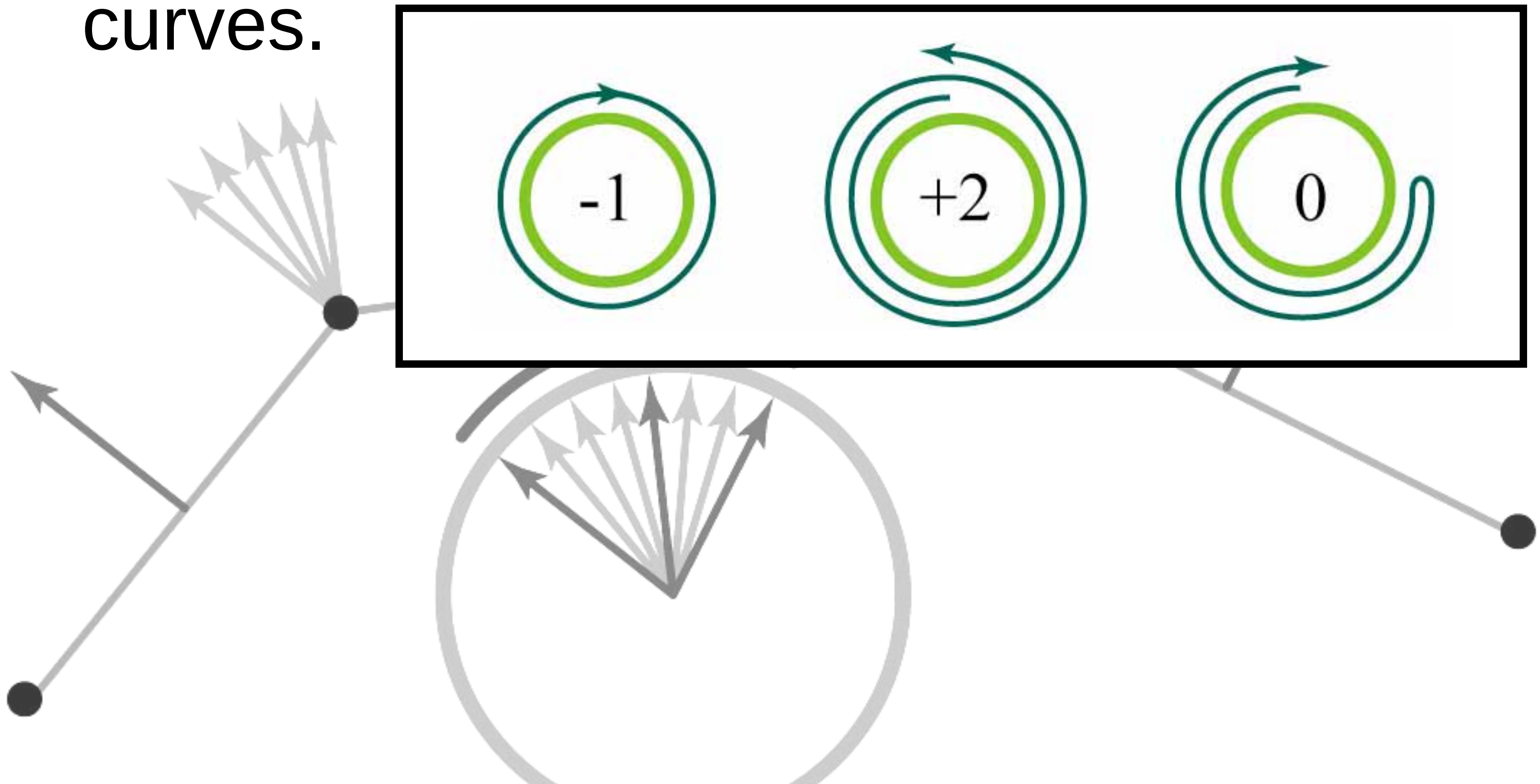
Discrete Gauß Map

Edges map to points, vertices map to arcs.



Discrete Gauß Map

Turning number well-defined for discrete curves.

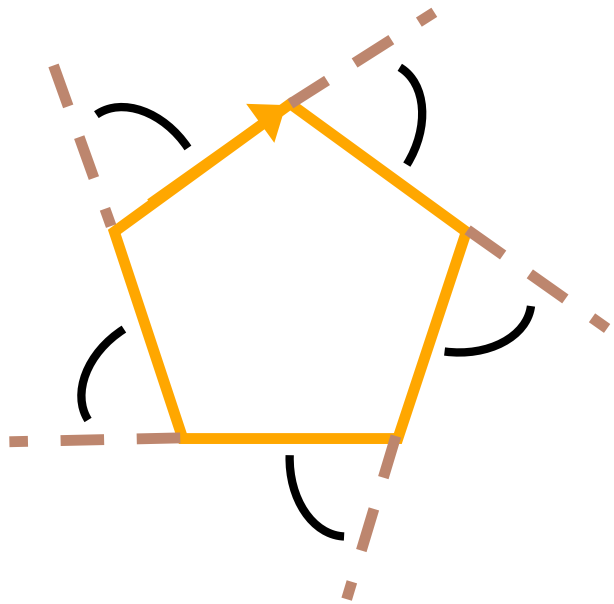


Discrete Turning Number Theorem

$$tsc(p) = \sum_{i=1}^{n} \alpha_i = 2\pi k$$

For a closed curve,
the total signed curvature is an integer multiple of 2π.

- proof: sum of exterior angles



Structure-preservation

Arbitrary discrete curve

- total signed curvature obeys
  discrete turning number theorem
- even coarse mesh
- which continuous theorems to preserve?
  - that depends on the application
  - fast-forward to last lecture:
    – Euclidian motions? triangle mesh is fine
    – Conformal maps? use circle-based mesh

Structure-preservation

Arbitrary discrete curve

- total signed curvature obeys discrete turning number theorem

discrete analogue of continuous theorem

- even coarse mesh
- which continuous theorems to preserve?
  - that depends on the application
  - fast-forward to last lecture:
    - Euclidian motions? triangle mesh is fine
    - Conformal maps? use circle-based mesh

Convergence

Consider refinement sequence

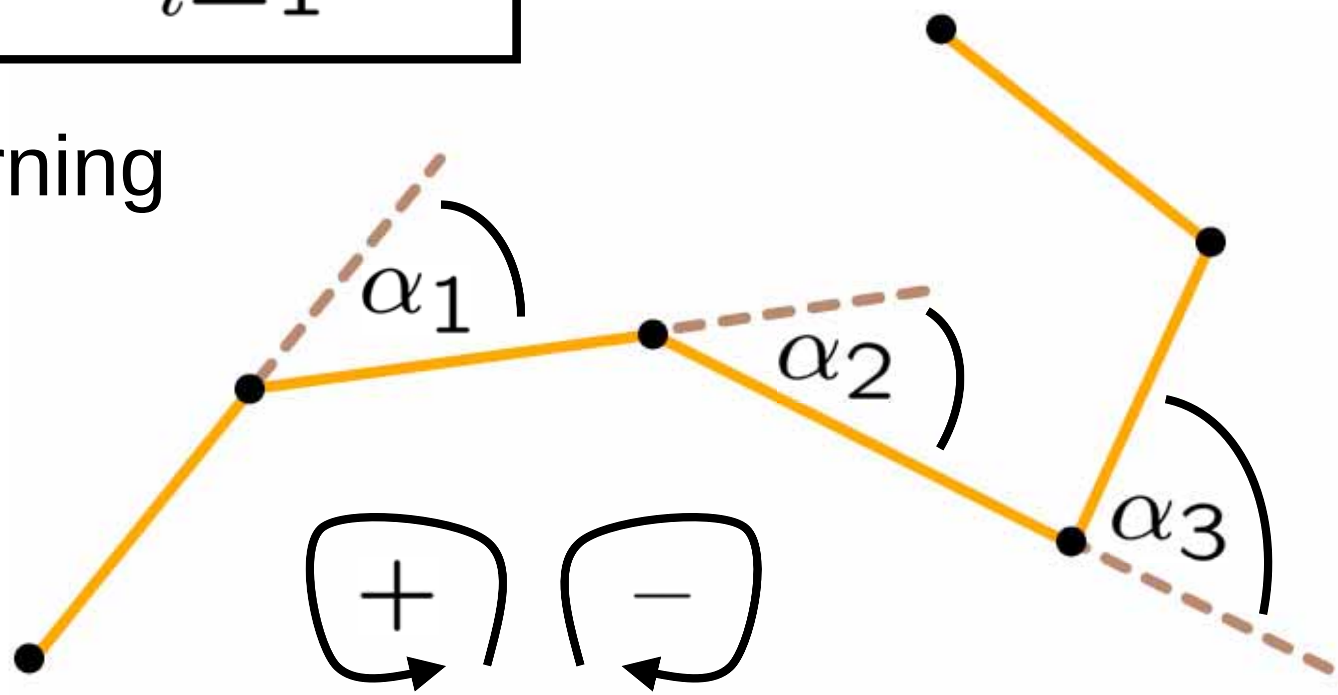
- length of inscribed polygon approaches length of smooth curve
- in general, discrete measure approaches continuous analogue
- which refinement sequence?
  - depends on discrete operator
  - pathological sequences may exist
- in what sense does the operator converge? (point-wise, L_2; linear,quadratic)

Recall:

Total signed curvature

$$tsc(p) = \sum_{i=1}^{n} \alpha_i$$

Sum of turning angles.

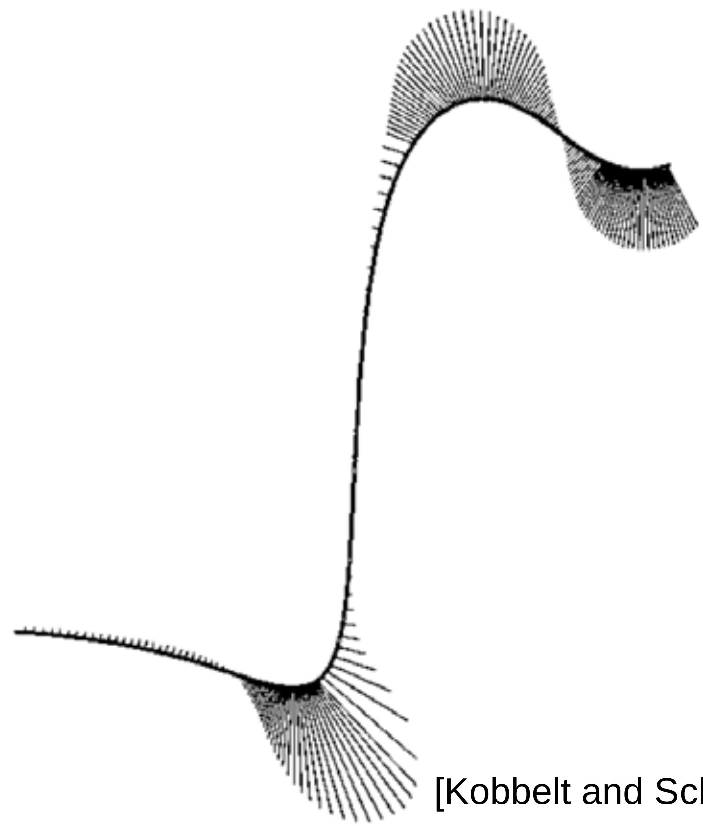


Other definitions for curvature

"Curvature normal"

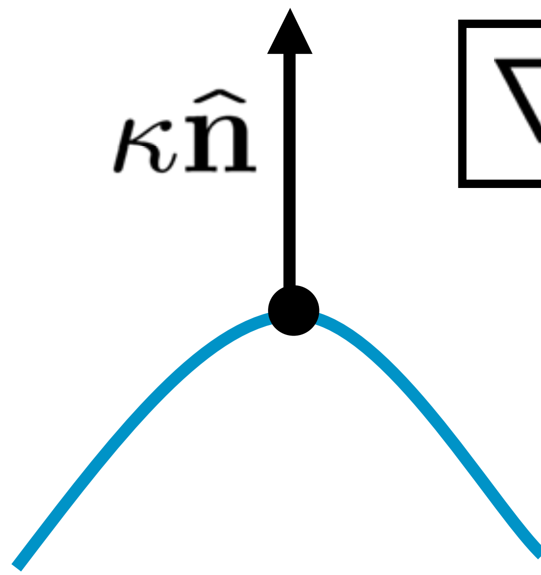
$$\kappa \hat{\mathbf{n}}$$

signed curvature (scalar) — unit normal (vector)



[Kobbelt and Schröder]

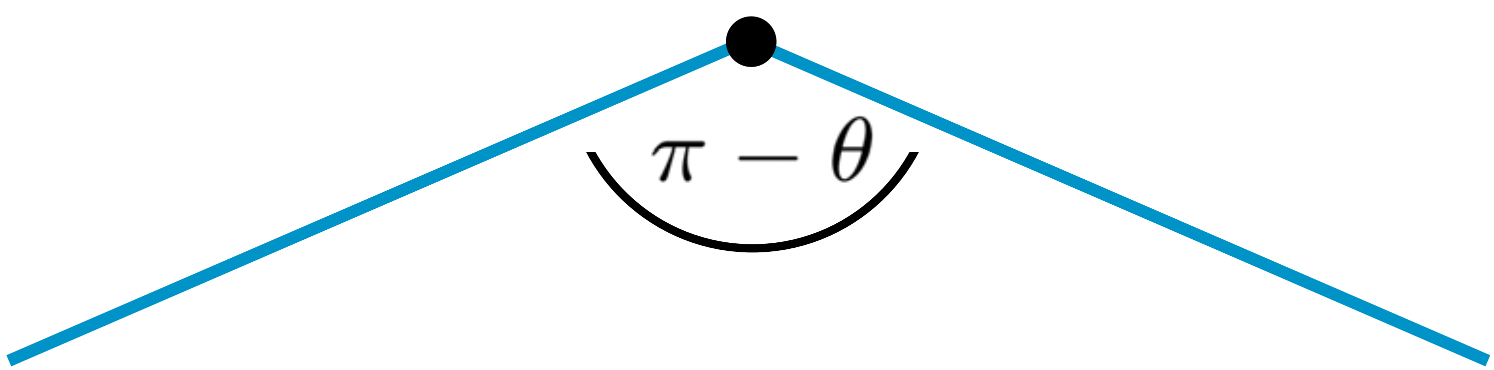
Curvature normal = length gradient



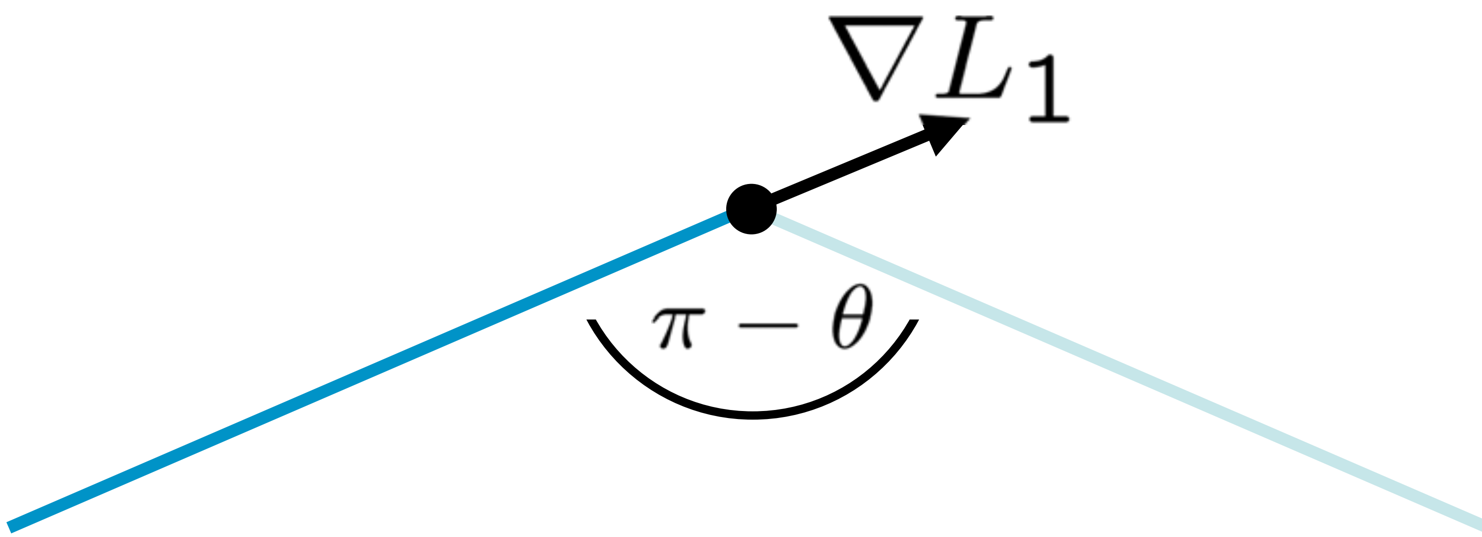
$$\boxed{\nabla L = \kappa\hat{\mathbf{n}}}$$

Use this to define discrete curvature!

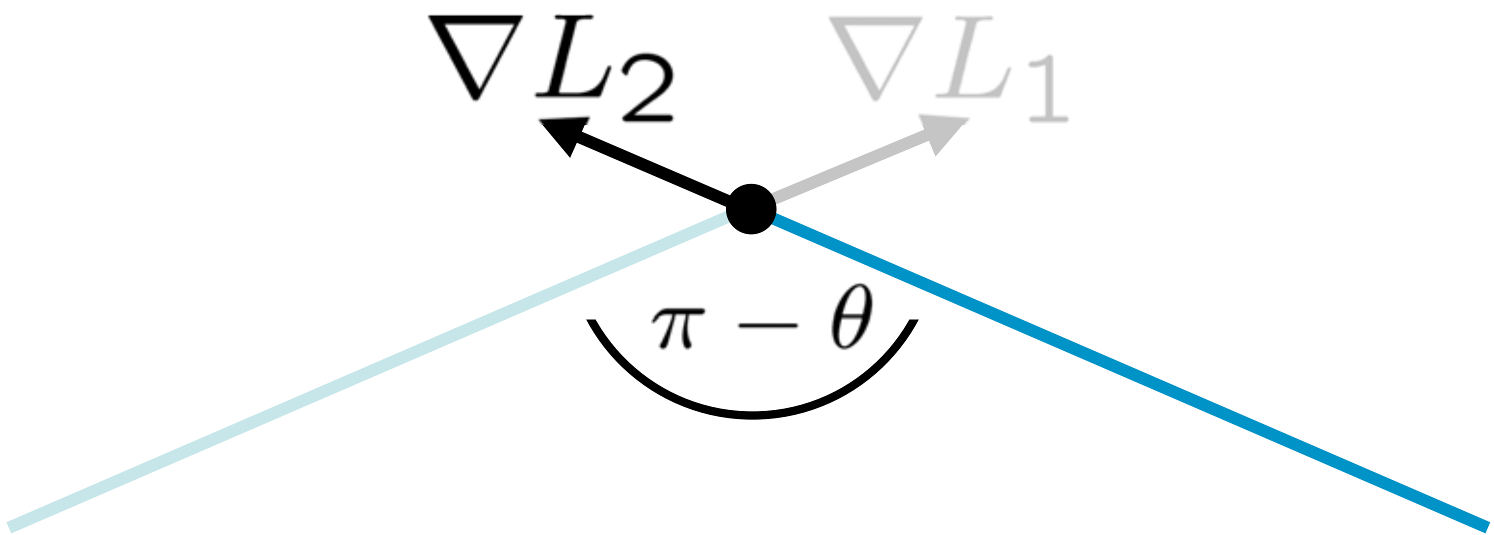
Curvature normal = length gradient



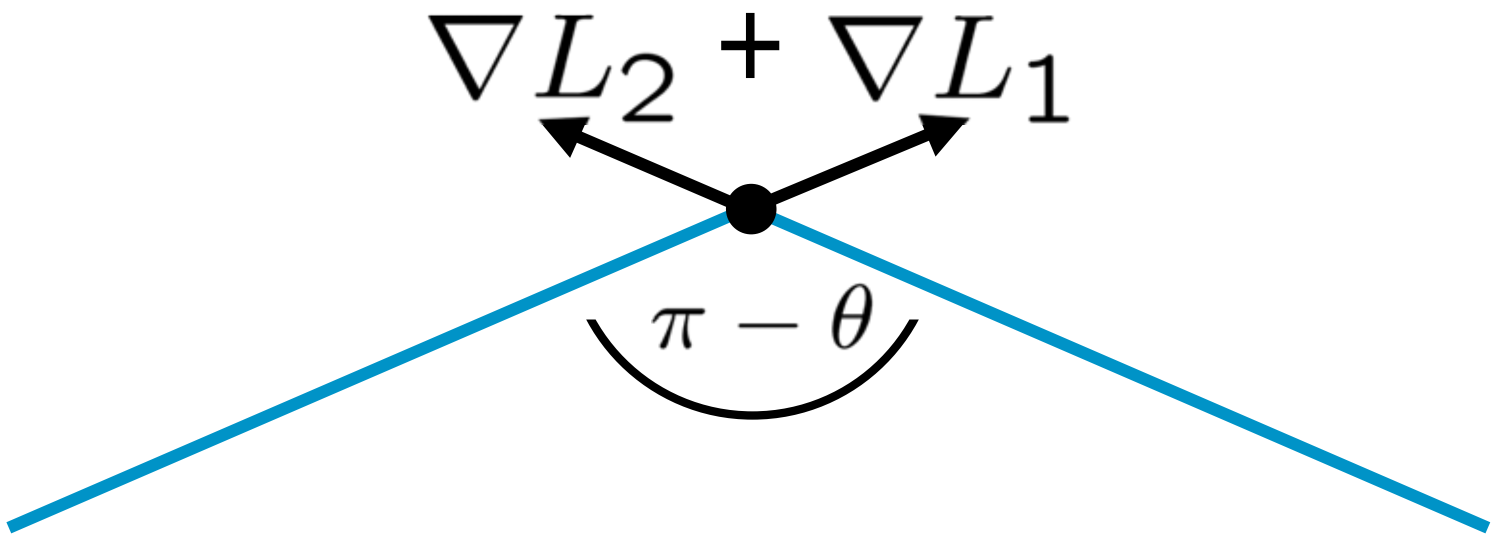
Curvature normal = length gradient



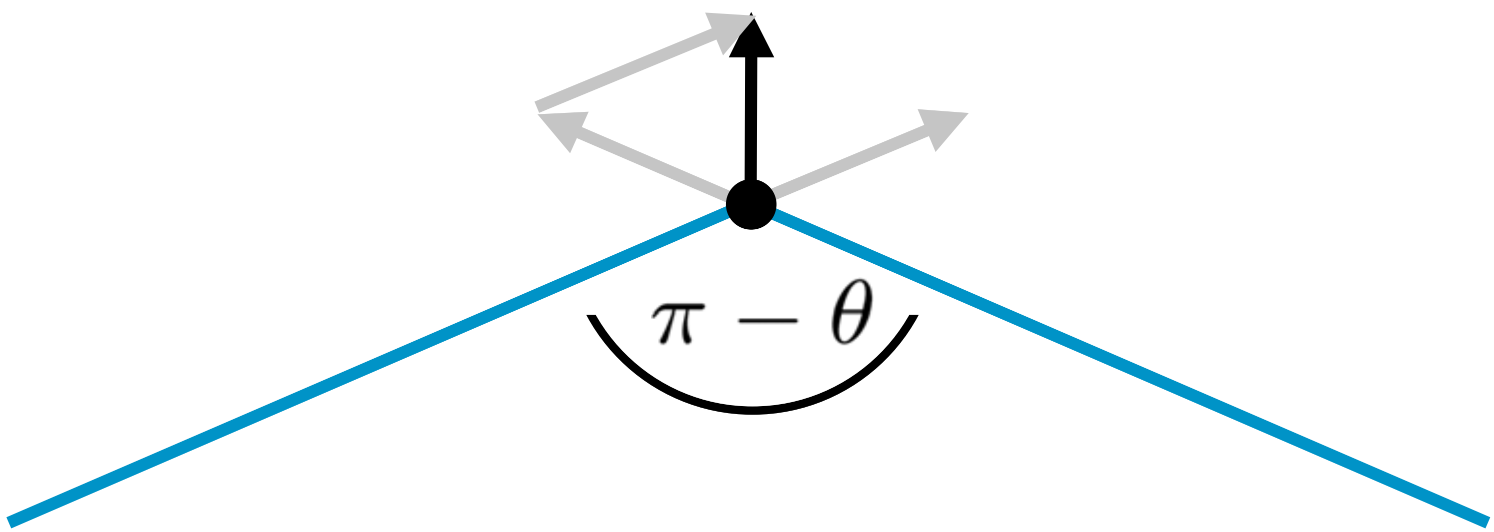
Curvature normal = length gradient



Curvature normal = length gradient

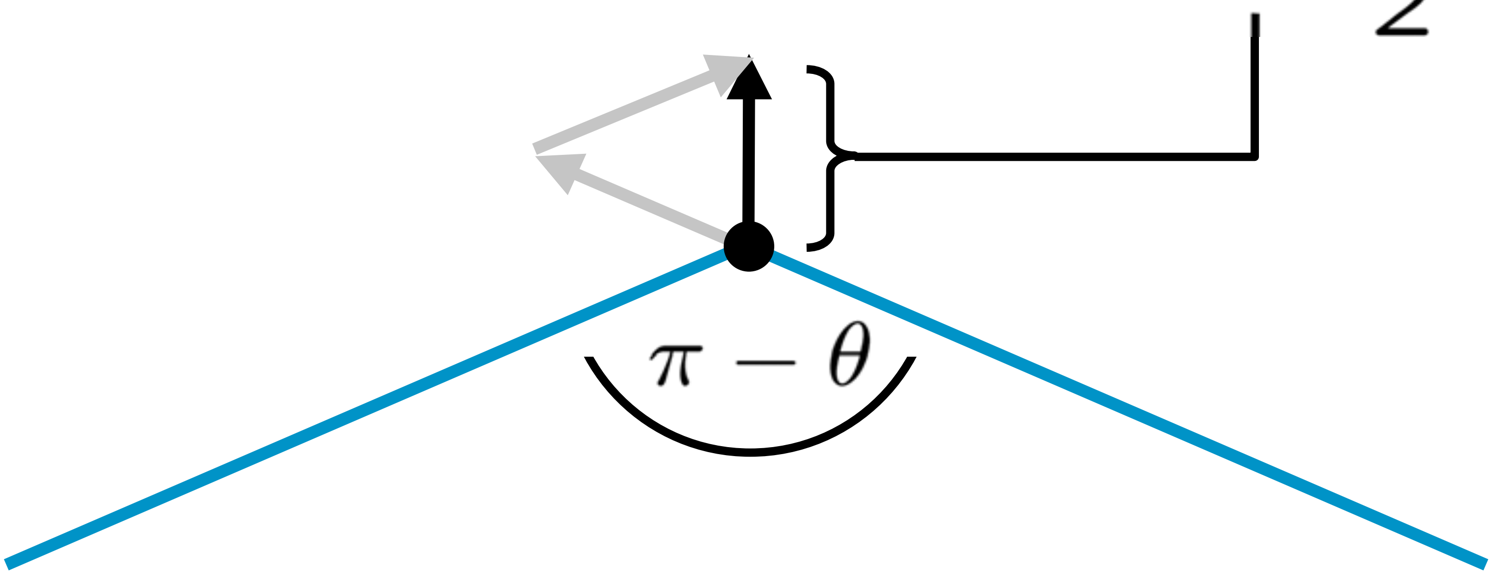


Curvature normal = length gradient



Curvature normal = length gradient

$$\nabla L = \kappa \hat{\mathbf{n}} = 2 \sin \frac{\theta}{2} \hat{\mathbf{n}}$$



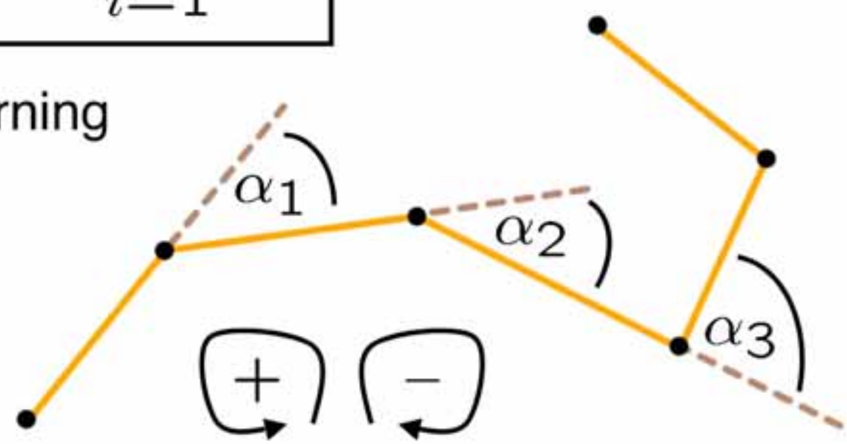
Curvature normal = length gradient

$$2\sin\frac{\theta}{2}\hat{\mathbf{n}}$$

Total signed curvature

$$tsc(p) = \sum_{i=1}^{n} \alpha_i$$

Sum of turning angles.



Recap

Structure-preservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature obeys the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature agree with continuous measures.