Invariant Measures of Convex Sets

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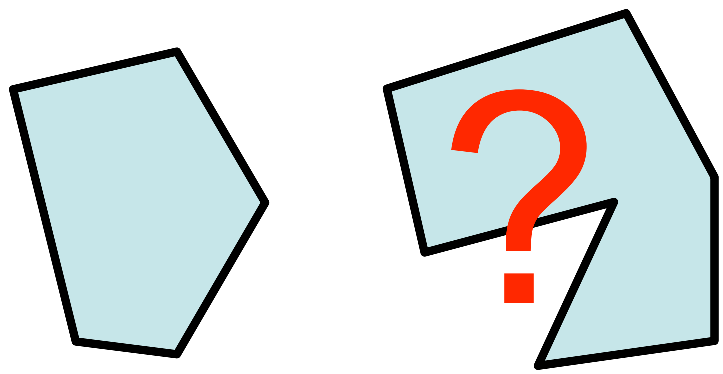
What will we measure?

Subject to be measured, S

$$S \subset \mathbb{R}^n$$

- an object living in n-dim space
- convex, compact subset of $\mathbb{R}^n$

- e.g.,
  - points, edges, faces



What will we measure?

$$S \subset \mathbb{R}^n$$

Subject to be measured, S

- an object living in n-dim space
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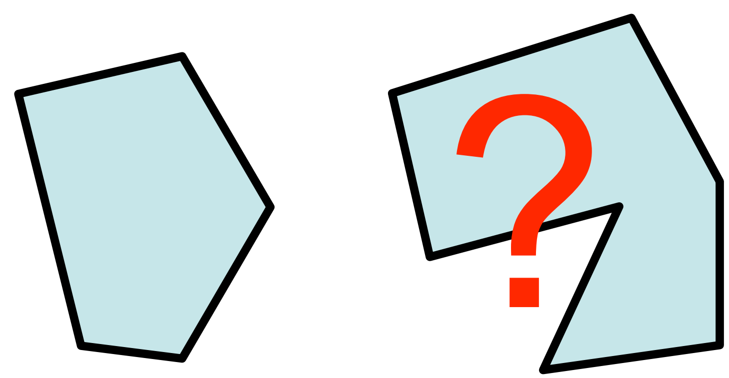
  CLOSED + BOUNDED

- e.g.,
  - points, edges, faces

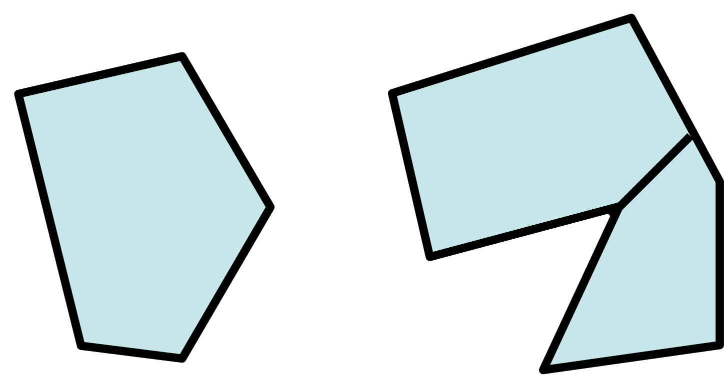


What will we measure?

Subject to be measured, S

$$\boxed{S \subset \mathbb{R}^n}$$

- an object living in n-dim space
- convex, compact subset of $\mathbb{R}^n$
  + finite unions and intersections
- e.g.,
  - points, edges, faces
  - a mesh



What is a reasonable measure?

Properties

- a measure is scalar-valued $\mu(S) \in \mathbb{R}$
- empty set $\mu(\emptyset) = 0$
- additivity $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$
- normalization (parallelotope, P)

What is a reasonable measure?

real number,
coordinate-frame invariant

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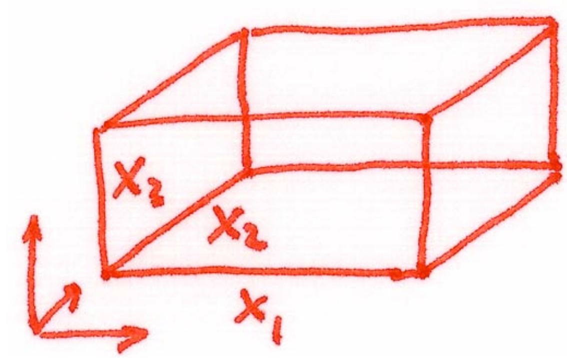
- normalization (parallelotope, P)
  - example: *volume*

$$\mu_n(P) = x_1 x_2 x_3 \ldots x_n$$



Other measures?

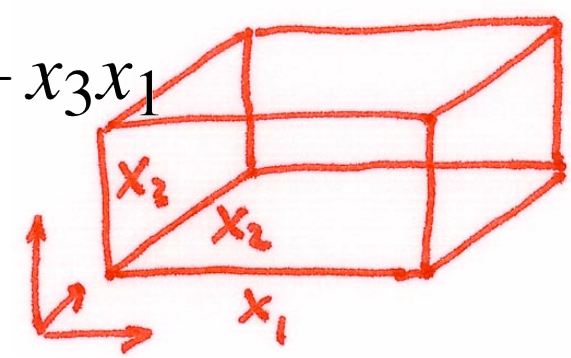
Other Measures

Elementary symmetric functions

$$
\begin{aligned}
e_1(x_1, x_2, \ldots, x_n) &= x_1 + x_2 + \ldots + x_n \\
e_2(x_1, x_2, \ldots, x_n) &= x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n \\
e_{n-1}(x_1, x_2, \ldots, x_n) &= \sum_i \prod_{j \neq i} x_j
\end{aligned}
$$

area/2 $\quad \mu_2(P) = x_1 x_2 + x_2 x_3 + x_3 x_1$

length/4 $\quad \mu_1(P) = x_1 + x_2 + x_3$



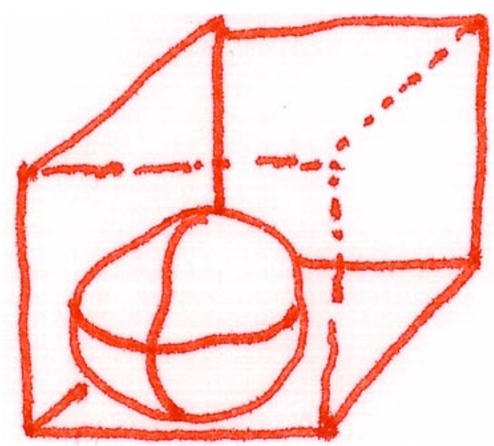
Invariant Measures

Intrinsic volumes

- n measures in n dimensions
- how to generalize to compact convex sets?

Geometric probability

- measure points in set
- probability of hitting set



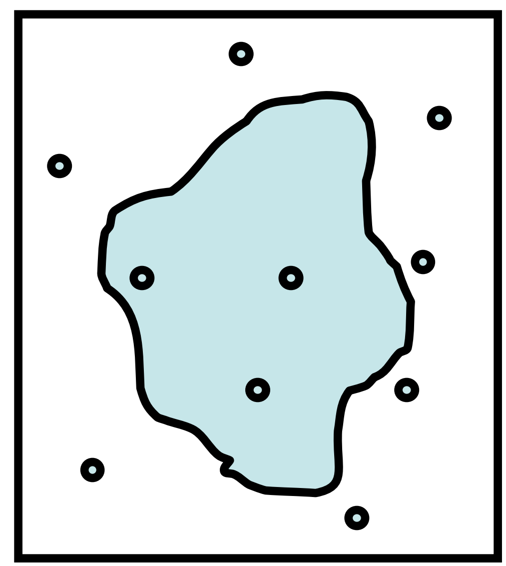
Geometric probability

Blindly throw darts… count number of hits

Darts: k-dim subspaces of n-D

- points
- lines
- planes
- volumes

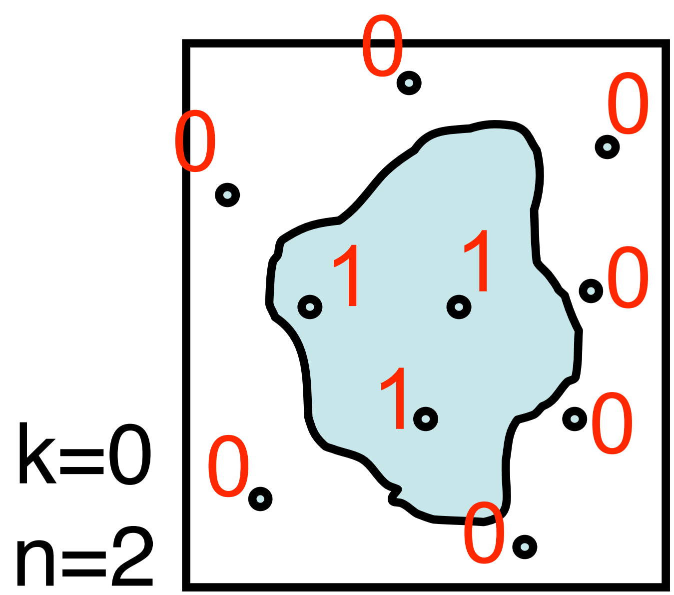
k=0
n=2



Geometric probability

Indicator function, $X_C(\omega_i)$

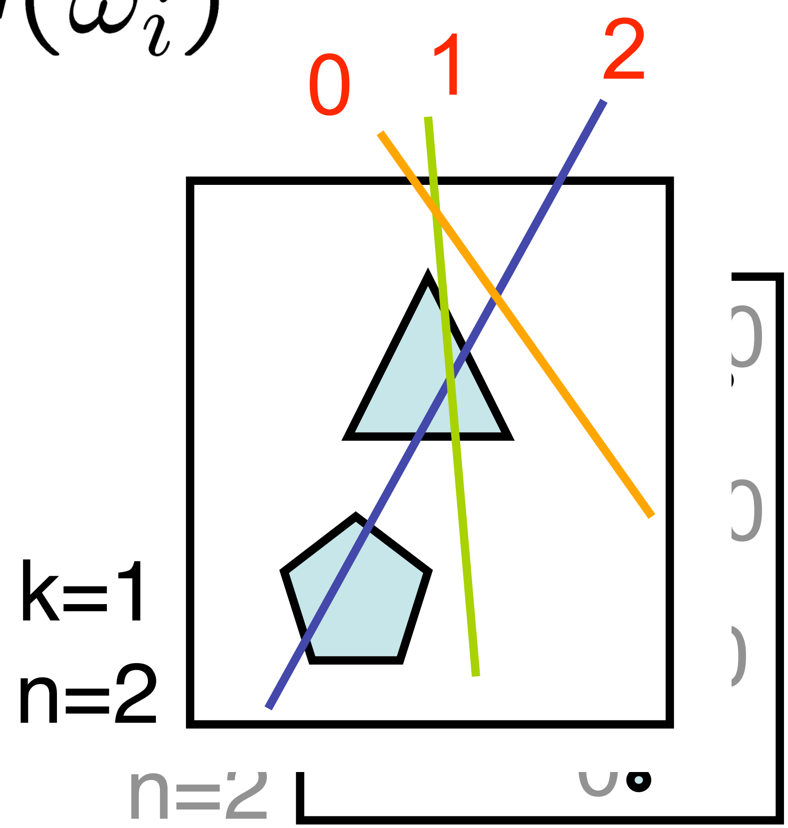
- input: a dart, ω
- output (point dart):
  1 if dart hits body
  0 if dart misses body



Geometric probability

Indicator function, $X_C(\omega_i)$

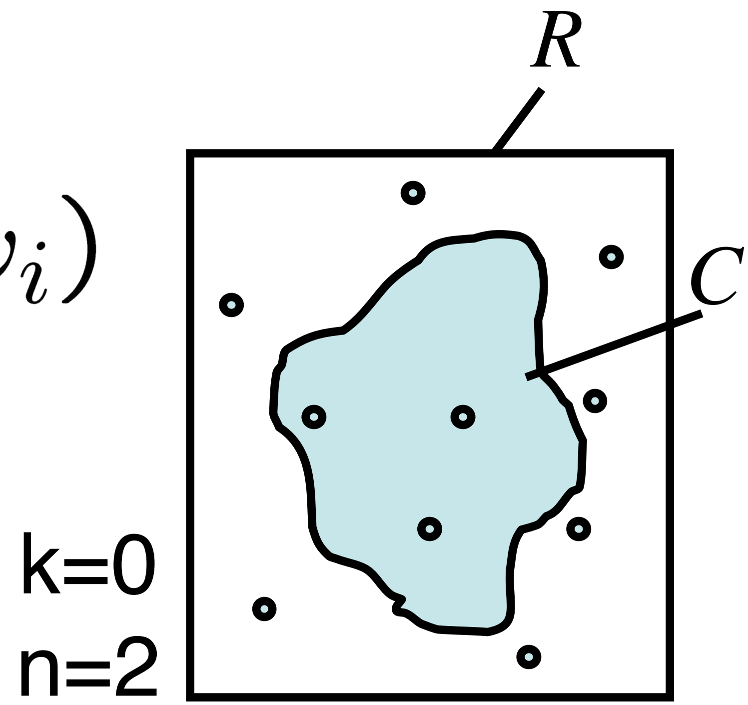
- input: a dart, ω
- output (point dart):
  1 if dart hits body
  0 if dart misses body
- in general,
  output is # hits



Geometric probability

Throw N random darts to estimate area

$$\frac{A_C}{A_R} \simeq \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$



Geometric probability

Throw N random darts to estimate area

Throw all the darts you have…

$$\frac{A_C}{A_R} \simeq \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$

$$A_C \propto \lim_{N->\infty} \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$

Geometric probability

Throw N random darts to estimate area

Throw all the darts you have…

2nd intrinsic volume of C

$\mu_2(C)$

measure of darts (k=0,n=2) hitting target C

$\lambda_0^2(C)$

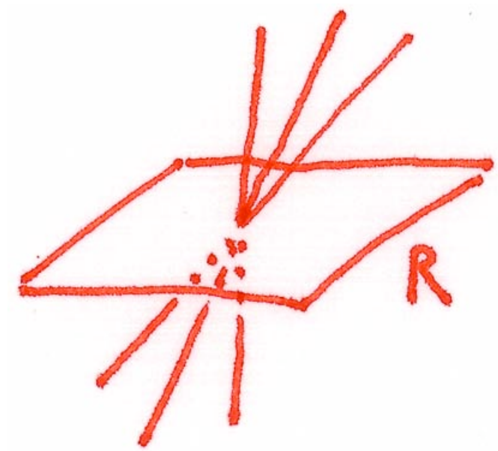
$$A_C \propto \boxed{\lim_{N->\infty} \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)}$$

Example: lines in $\mathbb{R}^3$

Measure of lines through rectangle, $\lambda_1^3(R)$

- prop. to area of R

$$\boxed{\lambda_1^3(R) \propto \mu_2(R)}$$



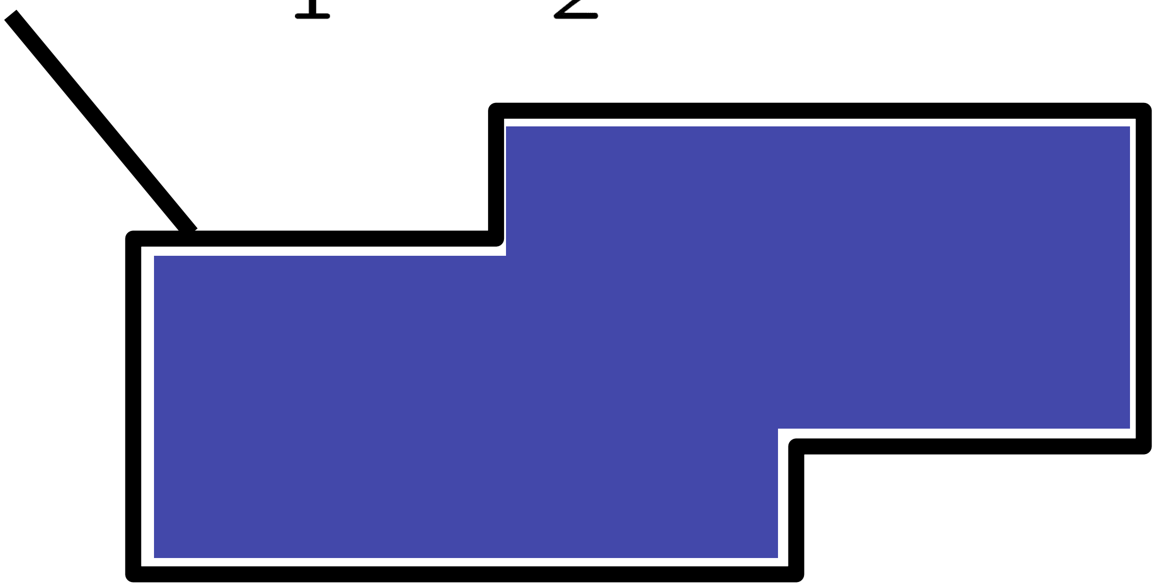
- two ways to explain
  - consider each fixed line orientation in turn
  - consider each point on the rectangle in turn

Example: lines in $\mathbb{R}^3$

Additivity in action

- consider the union of two rectangles

$$D = C_1 \cup C_2$$



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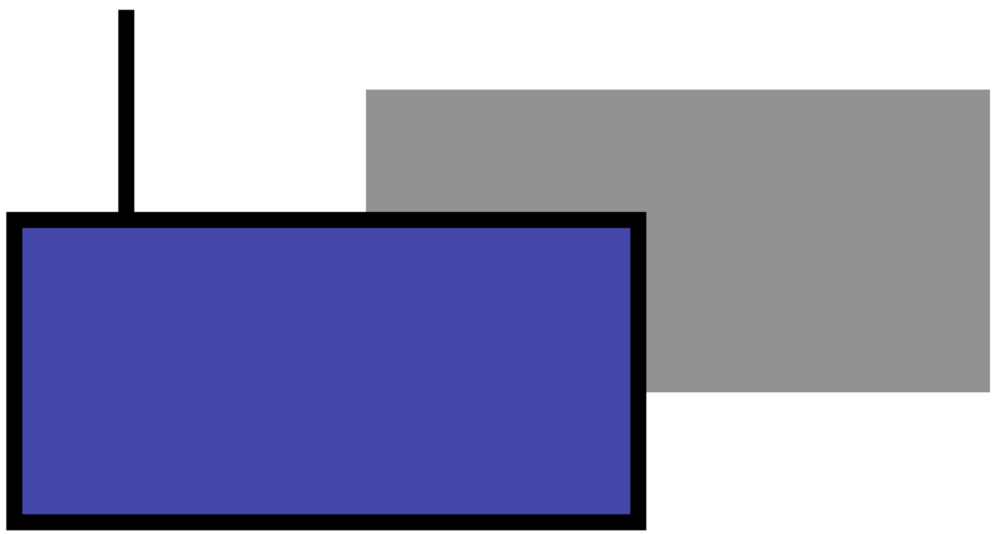

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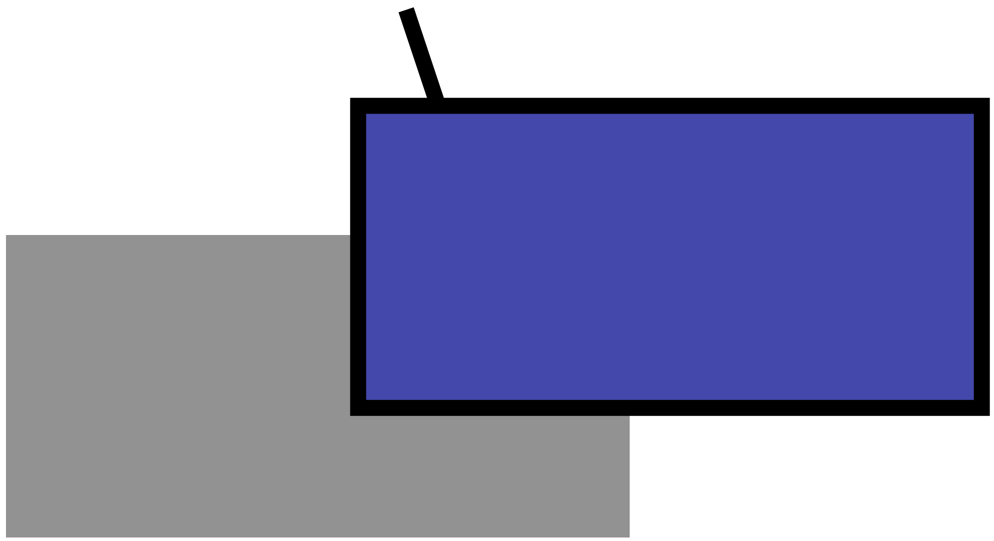

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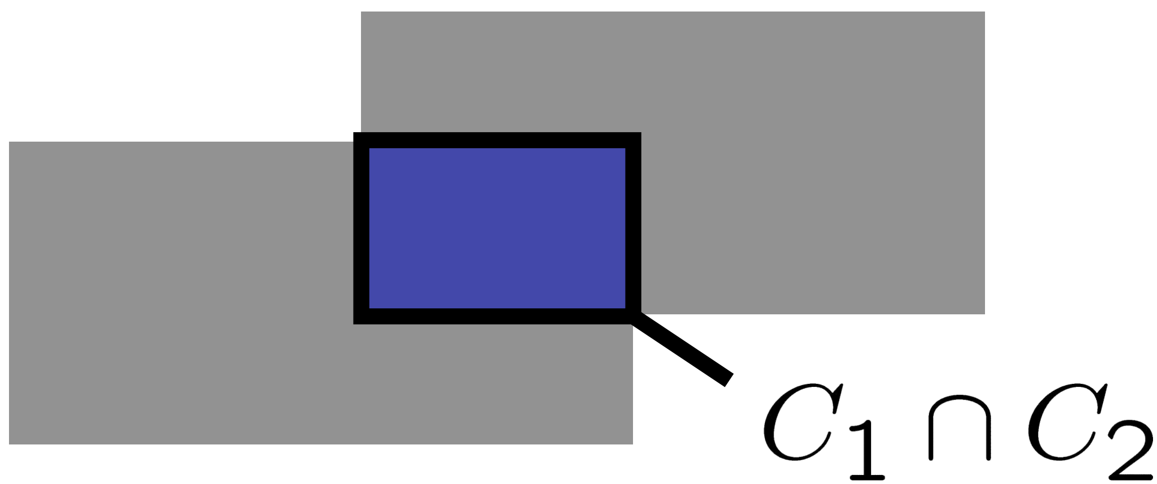
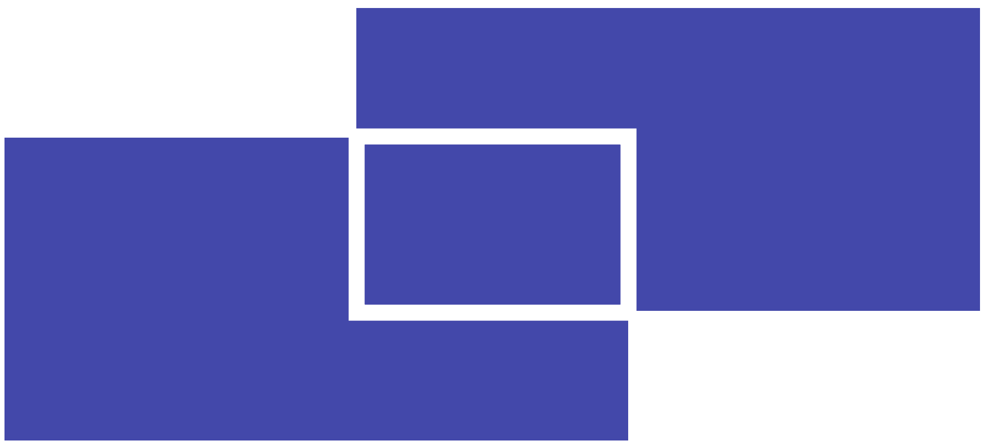


Example: lines in $\mathbb{R}^3$

Additivity in action

- consider the union of two rectangles
- inclusion - exclusion principle

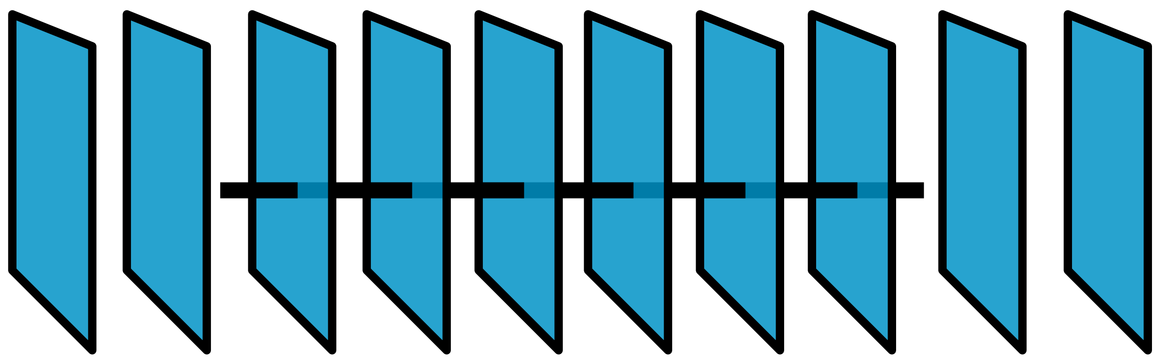
$$\mu_2(D) = \mu_2(C_1) + \mu_2(C_2) - \mu_2(C_1 \cap C_2)$$



Example: planes in $\mathbb{R}^3$

Measure of planes along a curve, $\lambda_2^3(c)$

- consider simplest curve: line segment
- $\lambda_2^3(c)$ proportional to length



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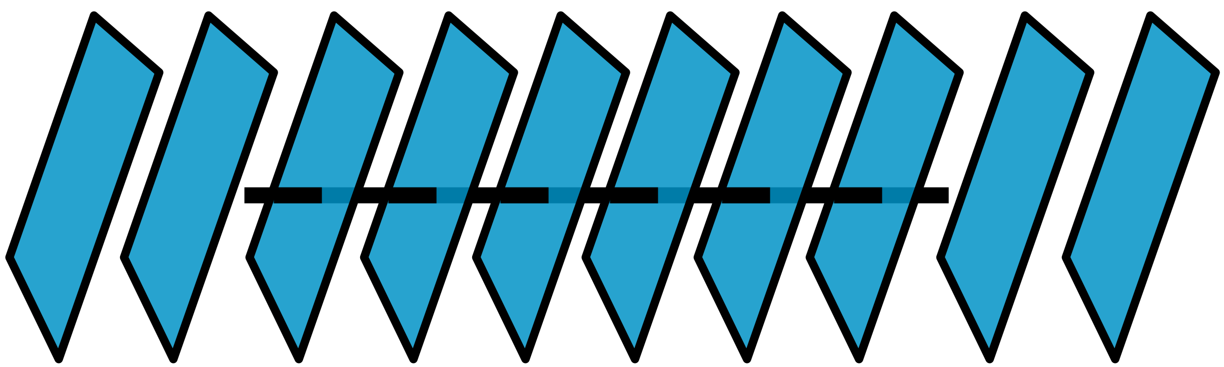
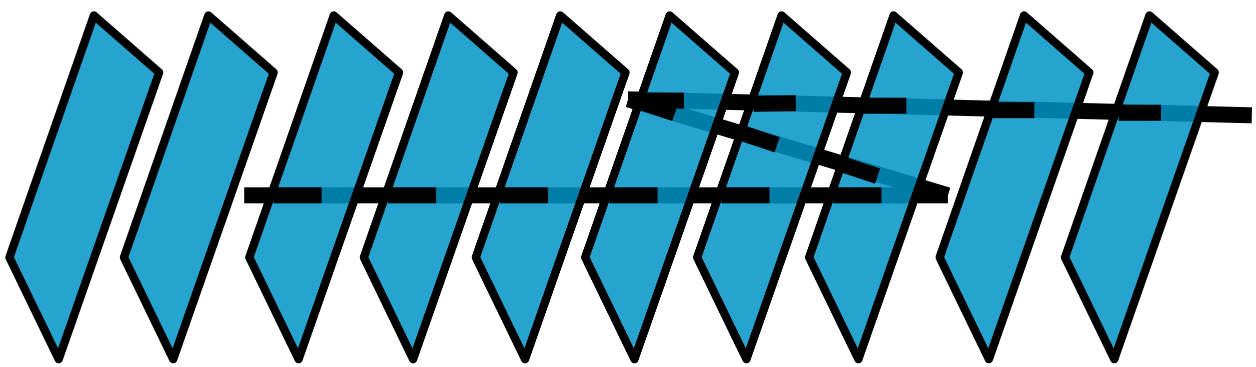


Example: planes in $\mathbb{R}^3$

Measure of planes along a curve, $\lambda_2^3(c)$

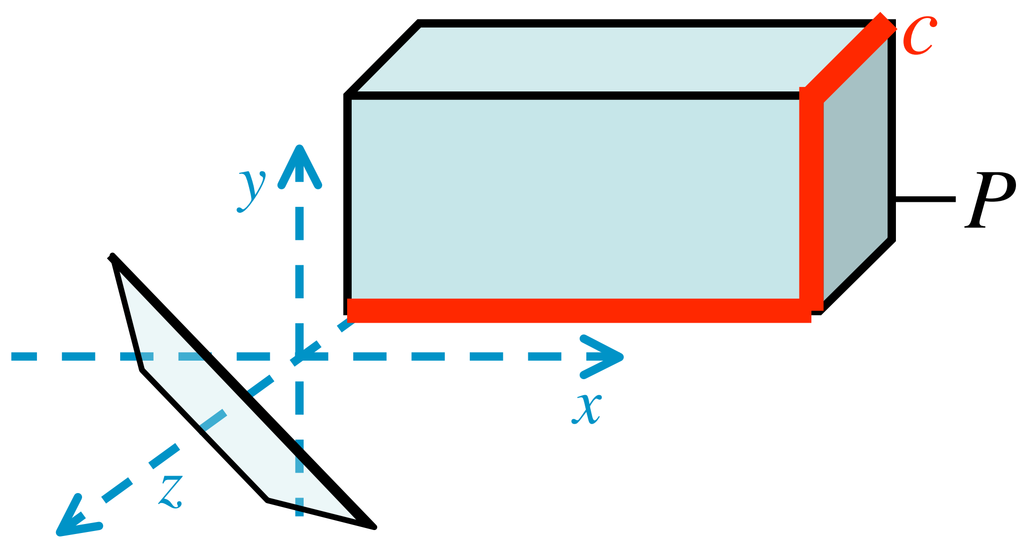
- consider polyline
- $\lambda_2^3(c)$ proportional to length



Example: planes in $\mathbb{R}^3$

Measure of planes hitting parallelotope, $\lambda_2^3(P)$

- consider "particularly useful" polyline
  - indicator fn. for c = indicator fn. for P



Example: planes in $\mathbb{R}^3$

Measure of planes hitting parallelotope, $\lambda_2^3(P)$

- consider "particularly useful" polyline
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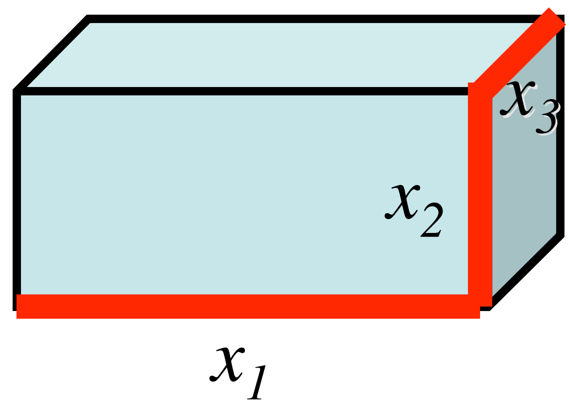
- $\lambda_2^3(c)$ proportional to length of polyline

$= x_1 + x_2 + x_3 = \mu_1(P)$

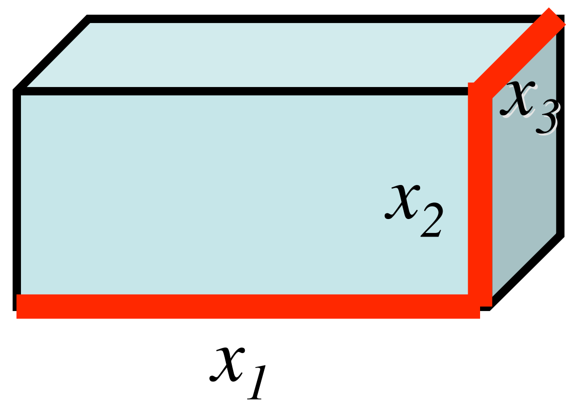


Example: planes in $\mathbb{R}^3$

Measure of planes hitting parallelotope, $\lambda_2^3(P)$

- consider "particularly useful" polyline
  - indicator fn. for c = indicator fn. for P
- $\lambda_2^3(c)$ proportional to length of polyline
  $= x_1 + x_2 + x_3 = \mu_1(P)$
- for general shape E,

$$\boxed{\int X_E(\omega)\, d\lambda_2^3(\omega) = \mu_1(E)}$$



Recap

Axioms

- it is a measure $\mu(\emptyset) = 0 \quad \mu(S) \in \mathbb{R}$
- Euclidian motion invariance
- additivity $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$
- some normalization
- μ_k(C) for compact convex set is the measure of all linear varieties of dim. n-k meeting C

But wait…

Do we have all of them?

- there is one missing
- recall: *elementary symmetric functions*

$$
\begin{aligned}
e_1(x_1, x_2, \ldots, x_n) &= x_1 + x_2 + \ldots + x_n \\
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But wait…

Do we have all of them?

- there is one missing
- recall: *elementary symmetric functions*
- what about $e_0(x_1,x_2,\ldots,x_n)$?

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$$

Euler characteristic

Symmetric function of order zero

- for any compact, convex set:

$$e_0 = \begin{cases} 1 & n > 0 \\ 0 & n = 0 \end{cases}$$

- “Euler characteristic”
- is this a measure?

Euler characteristic

Symmetric function of order zero

- for any compact, convex set:

$$e_0 = \begin{cases} 1 & n > 0 \\ 0 & n = 0 \end{cases}$$

homework!

$$\boxed{\#V - \#E + \#T - \ldots}$$

- “Euler characteristic χ”
- is this a measure?

Euler characteristic

Is it additive?

$$\boxed{\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)}$$

Euler characteristic

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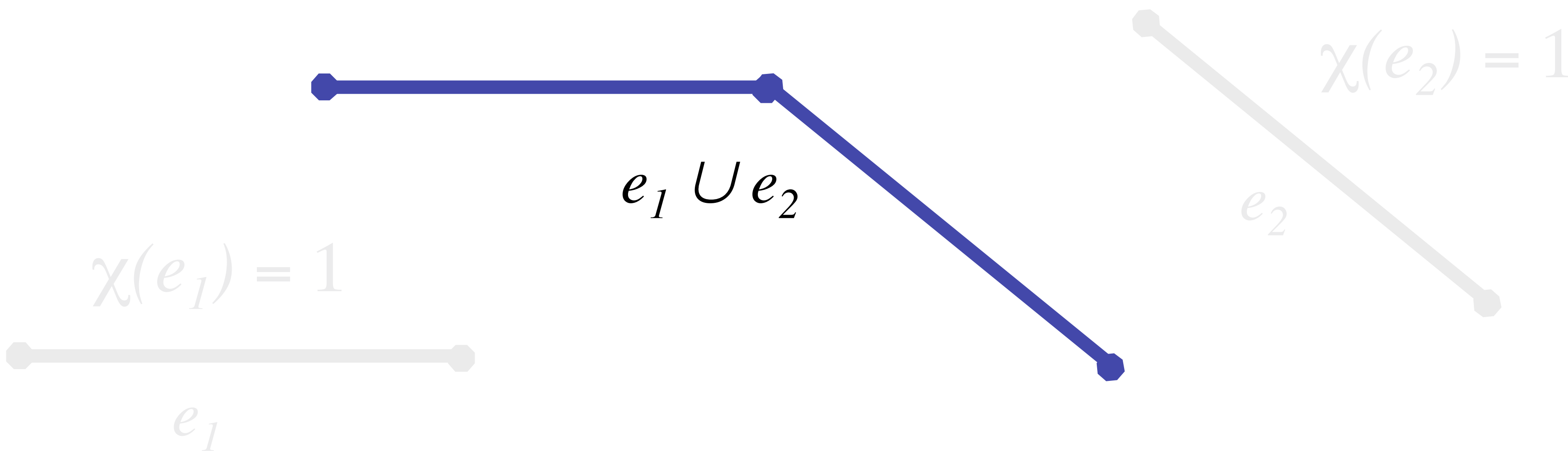

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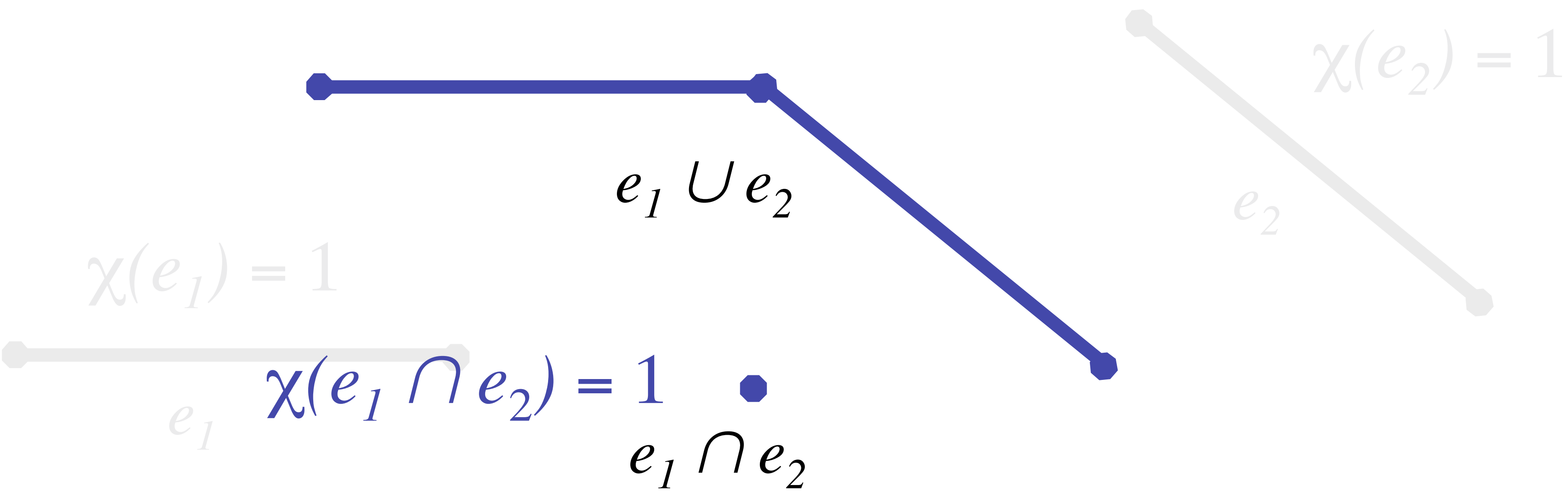

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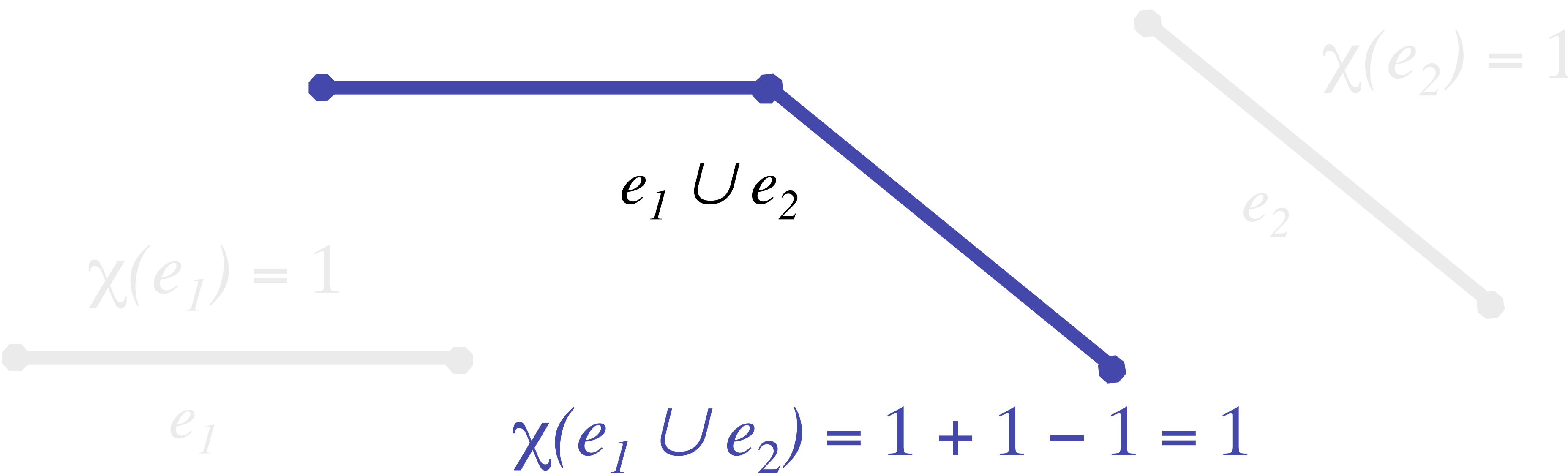


Euler characteristic

Is it additive? yes

Is it invariant under rigid-body motion? yes

Continuity

Our final axiom

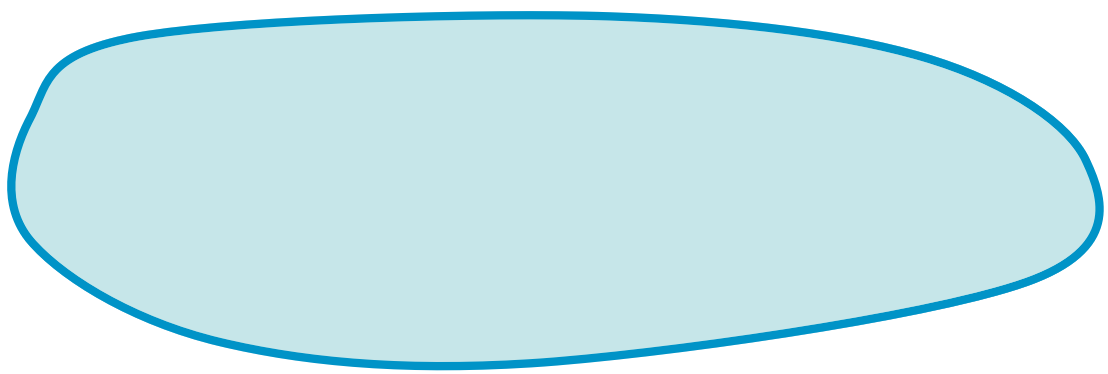
- measure should be continuous

$$\lim_{C_n \to C} \mu(C_n) = \mu(C)$$

Example: lines in $\mathbb{R}^3$

Measure of lines through planar surface

- limit process gives surface area for arbitrary planar surface (not just for R)



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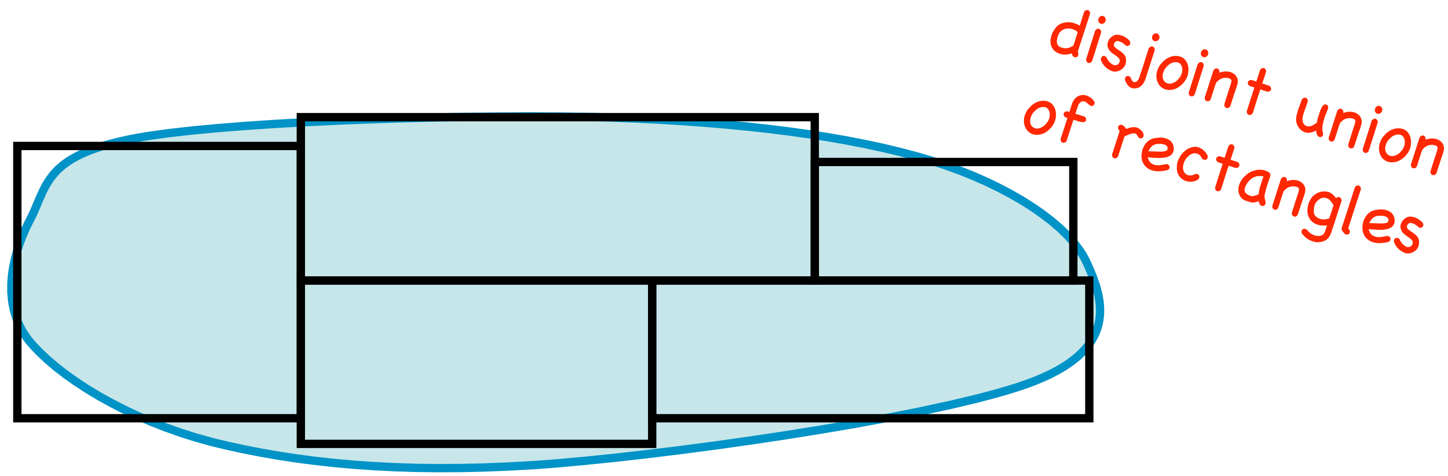

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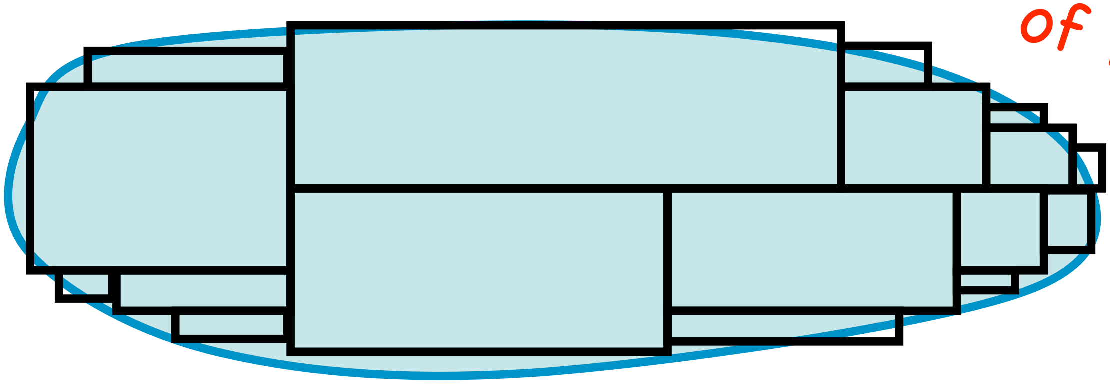

Example: lines in $\mathbb{R}^3$

Area of a (non-planar, disjoint) surface, D

1. partition into planar regions, C_i

disjoint union

$$D = \bigcup_{i=1}^{l} C_i$$

Example: lines in $\mathbb{R}^3$

Area of a (non-planar, disjoint) surface, D

1. partition into planar regions, C_i
2. add areas of each region

number of times ω meets D

$$\int X_D(\omega)\, d\lambda_1^3(\omega) = \sum_{i=1}^{l} \mu_2(C_i)$$

Example: lines in $\mathbb{R}^3$

Area of a (non-planar, disjoint) surface, D

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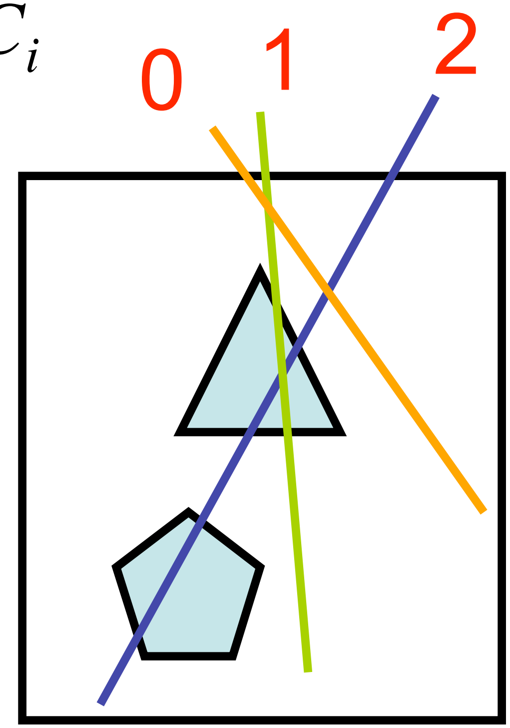

Here's the clincher...

Hadwiger (1957)

These measures
form a basis
for all continuous,
additive,
rigid motion invariant
measures on ring of convex sets.

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FUNDAMENAL RESULT

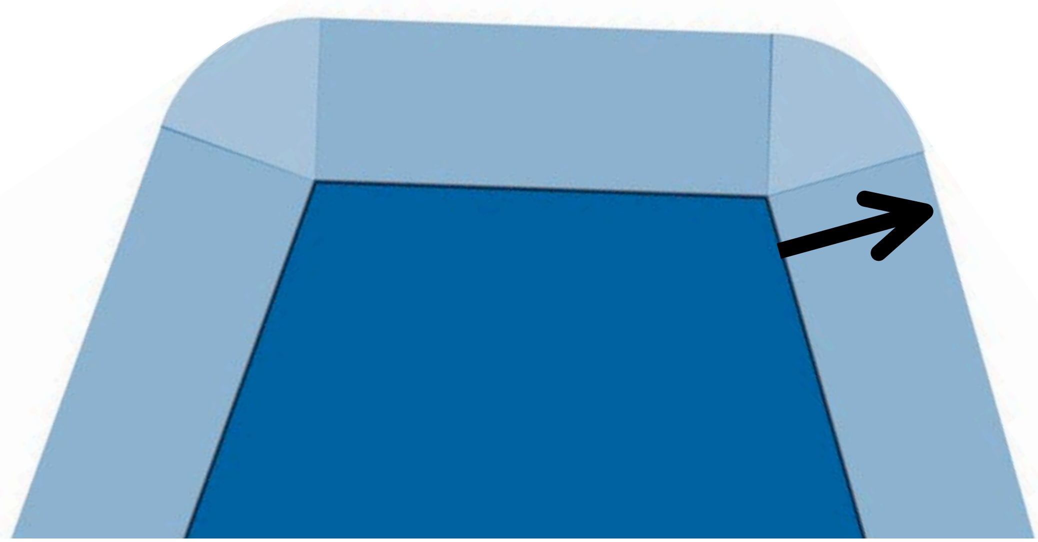
Putting it together

Steiner, Cauchy, Hadwiger

- expand a convex set outward by epsilon

min distance to set

$$\mathcal{K}_\varepsilon = \{x \in \mathbb{R}^n : d(\mathcal{K}, x) \leq \varepsilon\}$$



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- expand a convex set outward by epsilon

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$$\mathcal{K}_\varepsilon = \{x \in \mathbb{R}^n : d(\mathcal{K}, x) \leq \varepsilon\}$$

- the increase in volume is a polynomial in ε
- the coefficients are the intrinsic volumes!

Steiner example in $\mathbb{R}^3$

$$
\begin{aligned}
V_3(\mathcal{K}_\varepsilon) &= k_0 V_3(\mathcal{K}) && \text{volume} \\
&+ \varepsilon k_1 V_2(\mathcal{K}) && \text{area} \\
&+ \varepsilon^2 k_2 V_1(\mathcal{K}) && \text{mean width} \\
&+ \varepsilon^3 k_3 V_0(\mathcal{K}) && \text{Euler charac.}
\end{aligned}
$$

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Rewrite as boundary integral:
symmetric even f'n of principal curvatures:

$$\boxed{H_0=1,\ H_1=(\kappa_1+\kappa_2),\ H_2=\kappa_1\kappa_2}$$

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$$\int_{\partial\mathcal{K}} H_0 \, dA$$

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$$\int_{\partial \mathcal{K}} H_0 dA$$ = surface area

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= surface area

$$\int_{\partial\mathcal{K}} H_0 dA$$

$$\int_{\partial\mathcal{K}} H_1 dA$$

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Steiner example in $\mathbb{R}^3$

$$\begin{aligned} V_3(\mathcal{K}_\varepsilon) &= k_0 V_3(\mathcal{K}) \\ &+ \varepsilon k_1 V_2(\mathcal{K}) \\ &+ \varepsilon^2 k_2 V_1(\mathcal{K}) \\ &+ \varepsilon^3 k_3 V_0(\mathcal{K}) \end{aligned}$$

= surface area

$$\int_{\partial\mathcal{K}} H_0 \, dA$$

= total mean curvature

$$\int_{\partial\mathcal{K}} H_1 \, dA$$

Rewrite as boundary integral:

symmetric even f'n of principal curvatures:

$$\boxed{H_0=1,\ H_1=(\kappa_1+\kappa_2),\ H_2=\kappa_1\kappa_2}$$

Steiner example in $\mathbb{R}^3$

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\begin{aligned}
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&+ \varepsilon k_1 V_2(\mathcal{K}) \\
&+ \varepsilon^2 k_2 V_1(\mathcal{K}) \\
&+ \varepsilon^3 k_3 V_0(\mathcal{K})
\end{aligned}
$$

= surface area

$$\int_{\partial\mathcal{K}} H_0 dA$$

= total mean curvature

$$\int_{\partial\mathcal{K}} H_1 dA$$

$$\frac{1}{3}\int_{\partial\mathcal{K}} H_2 dA$$

Rewrite as boundary integral:
symmetric even f'n of principal curvatures:

$$\boxed{H_0=1,\ H_1=(\kappa_1+\kappa_2),\ H_2=\kappa_1\kappa_2}$$

Steiner example in $\mathbb{R}^3$

$$\begin{aligned} V_3(\mathcal{K}_\varepsilon) &= k_0 V_3(\mathcal{K}) \\ &+ \varepsilon k_1 V_2(\mathcal{K}) \\ &+ \varepsilon^2 k_2 V_1(\mathcal{K}) \\ &+ \varepsilon^3 k_3 V_0(\mathcal{K}) \end{aligned}$$

= surface area

$$\int_{\partial\mathcal{K}} H_0 dA$$

= total mean curvature

$$\int_{\partial\mathcal{K}} H_1 dA$$

= Euler charac.

$$\frac{1}{3}\int_{\partial\mathcal{K}} H_2 dA$$

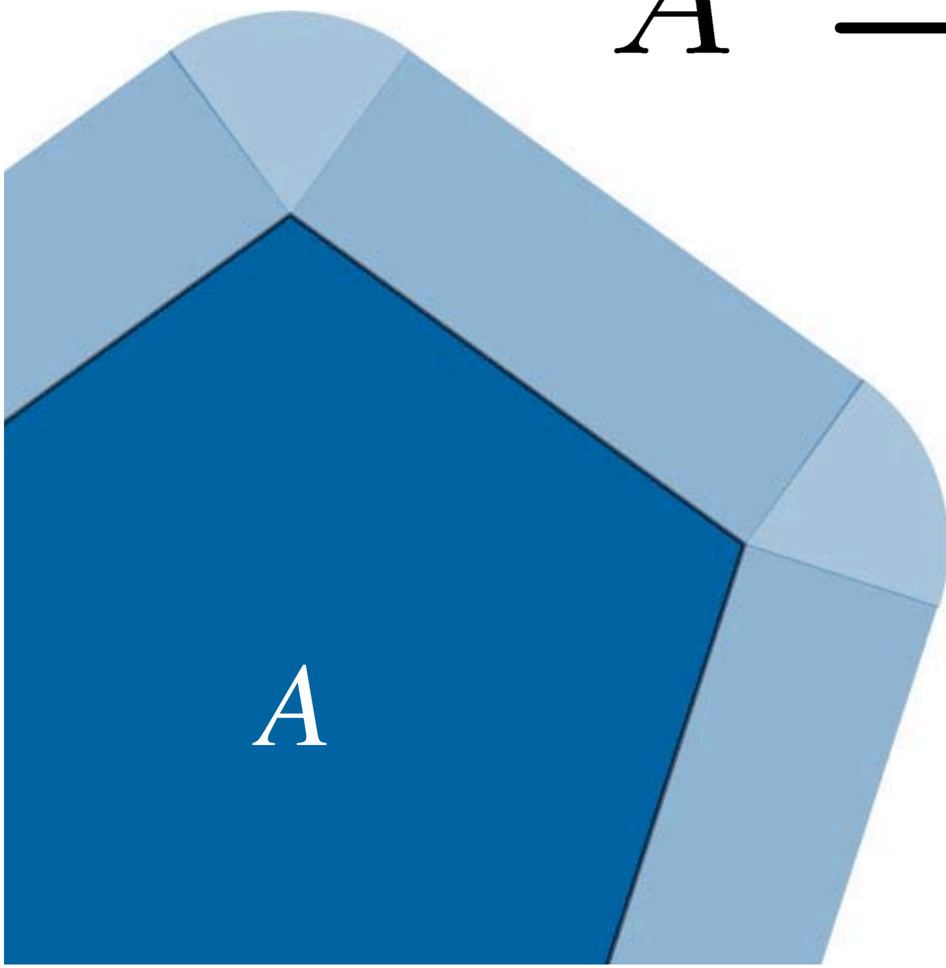
Rewrite as boundary integral:

symmetric even f'n of principal curvatures:

$\boxed{H_0=1,\ H_1=(\kappa_1+\kappa_2),\ H_2=\kappa_1\kappa_2}$

A Steiner walk-through, 2d

$$A' = A + \ldots$$



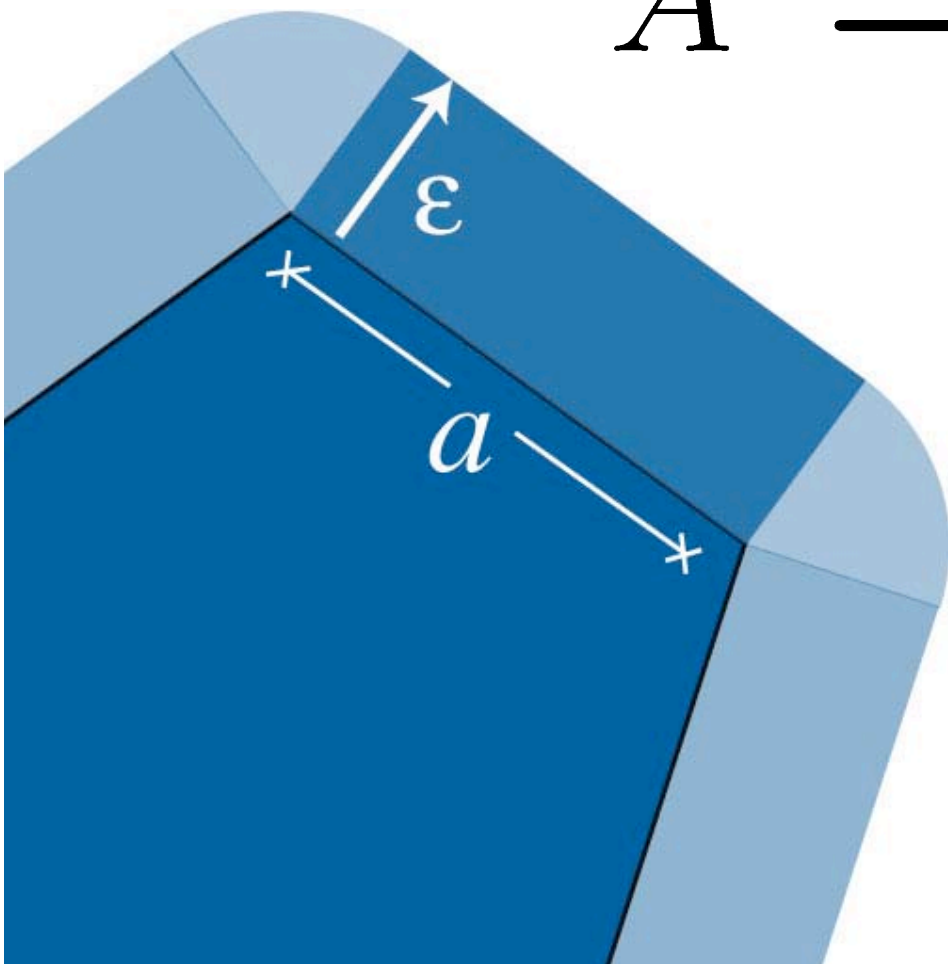
Inflate a planar polygon by epsilon

What is the new area?

A Steiner walk-through, 2d

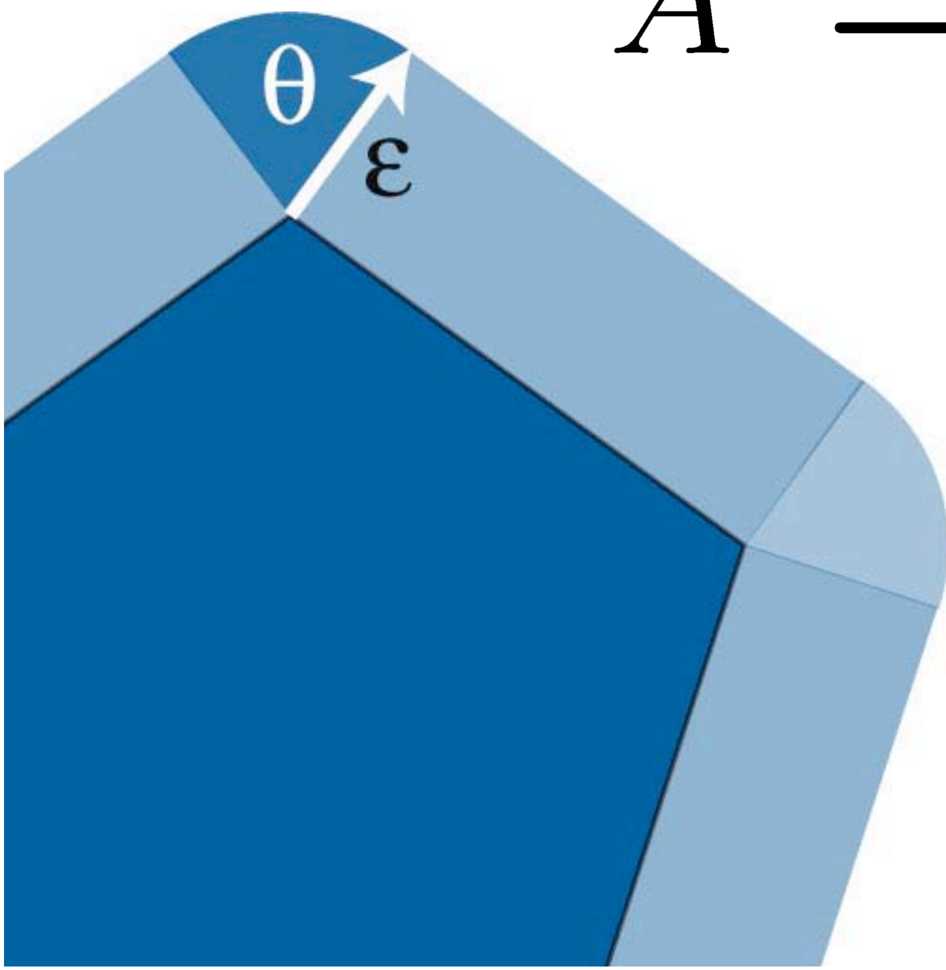
$$A' = A + \sum_i \epsilon a_i$$

$$+ \dots$$



Each edge contributes
a rectangle

A Steiner walk-through, 2d

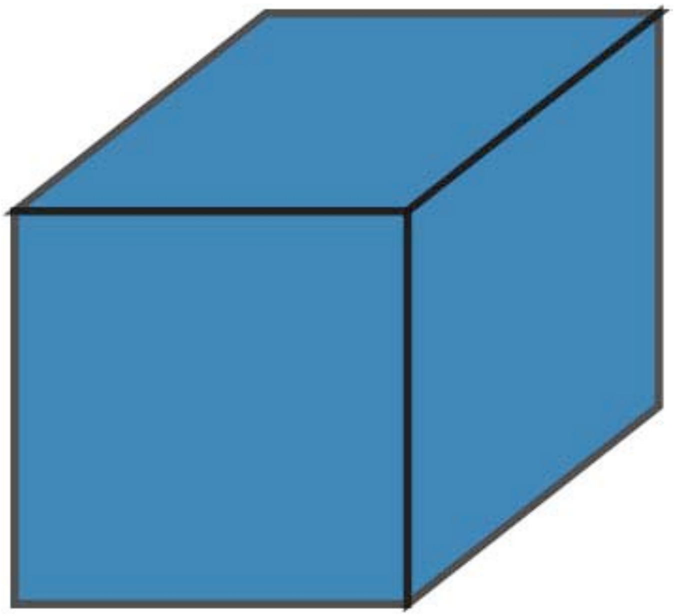


$$A' = A + \sum_i \epsilon a_i + \sum_j \pi \epsilon^2 \theta_j$$

Each vertex
contributes a sector

A Steiner walk-through, 3d

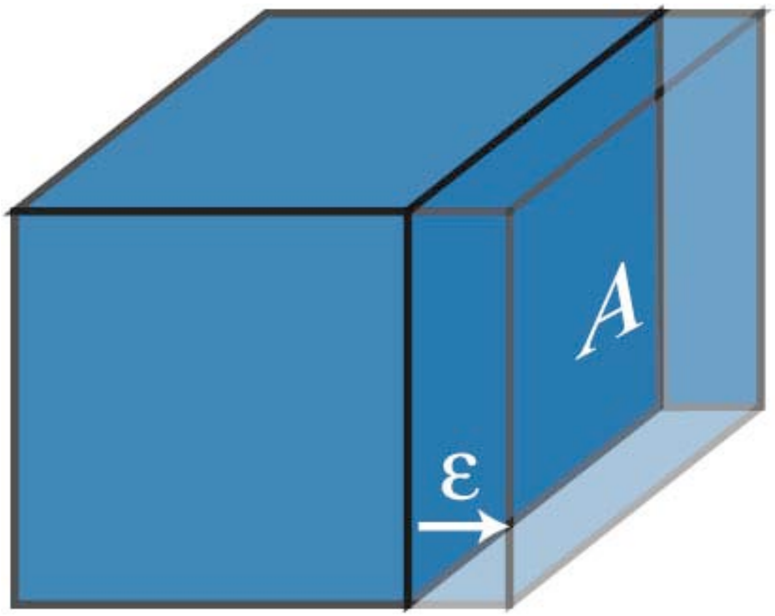
$$V' = V + \ldots$$



Inflate a polyhedron

What is the new volume?

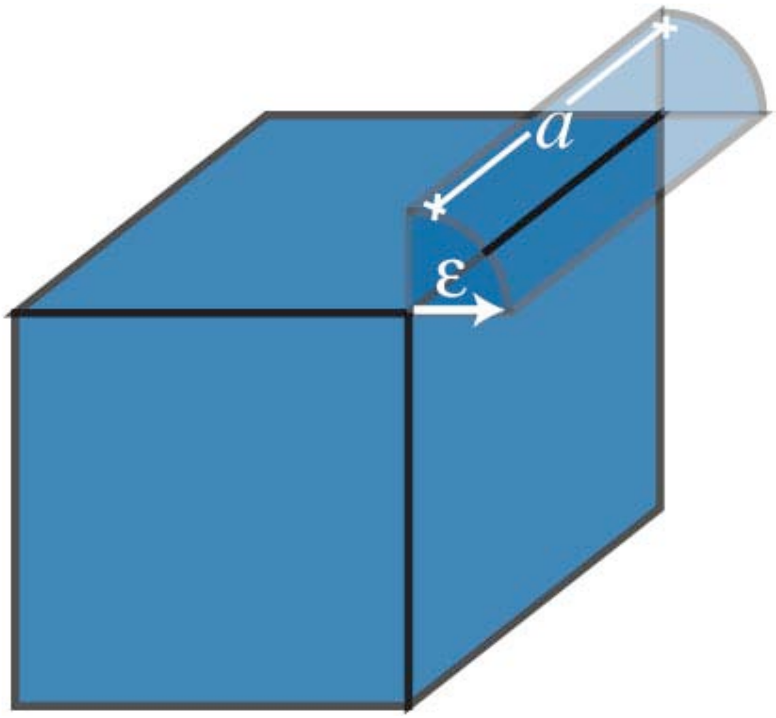
A Steiner walk-through, 3d



$$V' = V + \sum_i \epsilon A_i$$

Each face contributes a parallelotope

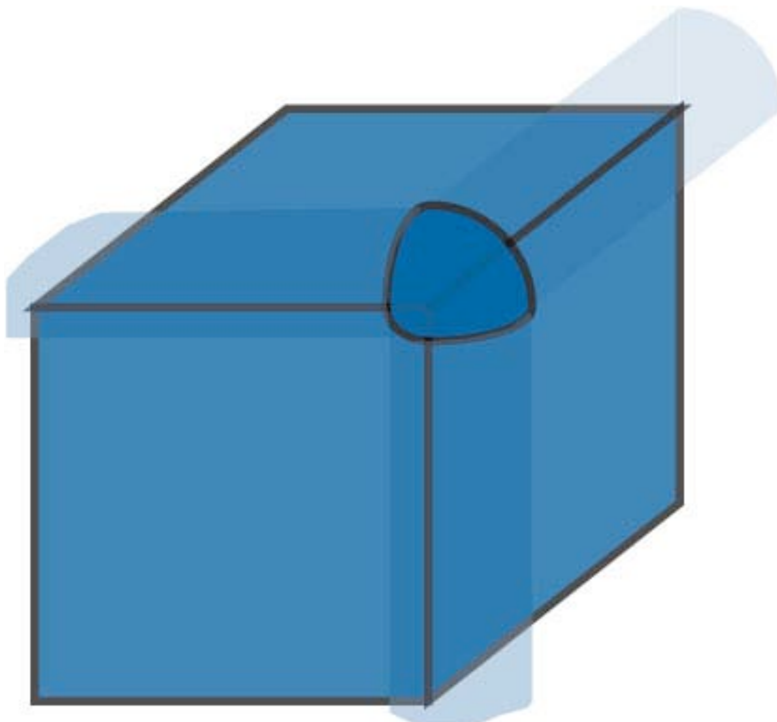
A Steiner walk-through, 3d



$$V' = V + \sum_i \epsilon A_i + \sum_j \epsilon^2 a\theta$$

Each edge contributes
a wedge of a cylinder

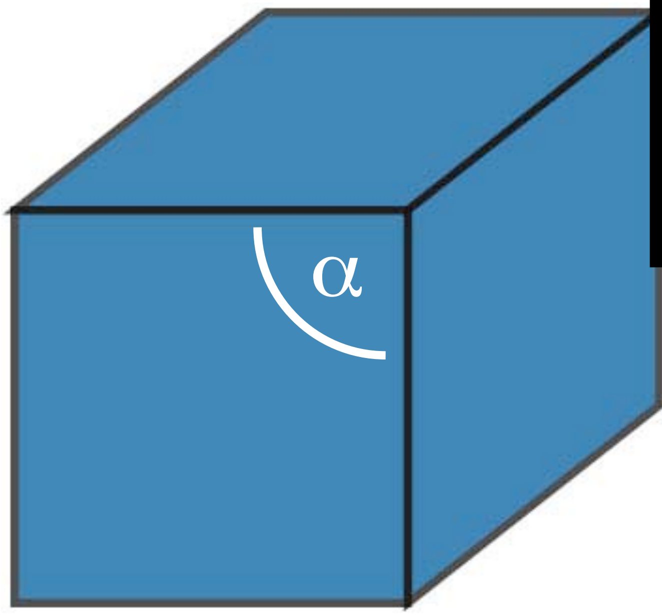
A Steiner walk-through, 3d



$$V' = V + \sum_i \epsilon A_i + \sum_j \epsilon^2 a\theta + \sum_l \epsilon^3 K_l$$

Each vertex contributes a spherical wedge

A Steiner walk-through, 3d



$$V' = V$$

$$\boxed{K = 2\pi - \sum_m \alpha_m}$$

$$+ \sum_j \epsilon^2 a\theta$$

$$+ \sum_l \epsilon^3 K_l$$

Each vertex contributes a spherical wedge

A Steiner walk-through, 3d

$$V' = V + \sum_i \epsilon A_i + \sum_j \epsilon^2 a\theta + \sum_l \epsilon^3 K_l$$

Steiner example in $\mathbb{R}^3$

$$\begin{aligned} V_3(\mathcal{K}_\varepsilon) &= k_0 V_3(\mathcal{K}) && \text{volume} \\ &+ \varepsilon k_1 V_2(\mathcal{K}) && \text{area} \\ &+ \varepsilon^2 k_2 V_1(\mathcal{K}) && \text{mean width} \\ &+ \varepsilon^3 k_3 V_0(\mathcal{K}) && \text{Euler charac.} \end{aligned}$$

Rewrite as boundary integral:
symmetric even f'n of principal curvatures:

$$\boxed{H_0=1,\ H_1=(\kappa_1+\kappa_2),\ H_2=\kappa_1\kappa_2}$$

A Steiner walk-through, 3d

$$
\boxed{
\begin{aligned}
V_3(\mathcal{K}_\varepsilon) &= k_0 V_3(\mathcal{K}) \\
&+ \varepsilon k_1 V_2(\mathcal{K}) \\
&+ \varepsilon^2 k_2 V_1(\mathcal{K}) \\
&+ \varepsilon^3 k_3 V_0(\mathcal{K})
\end{aligned}
}
$$

$$
\begin{aligned}
V' = V & \\
&+ \sum_i \epsilon A_i \\
&+ \sum_j \epsilon^2 a\theta \\
&+ \sum_l \epsilon^3 K_l
\end{aligned}
$$

A Steiner walk-through, 3d

$$V' = V + \sum_i \epsilon A_i + \sum_j \epsilon^2 a\theta + \sum_l \epsilon^3 K_l$$

$$\boxed{\begin{aligned} V_3(\mathcal{K}_\varepsilon) &= k_0 V_3(\mathcal{K}) \\ &+ \varepsilon k_1 V_2(\mathcal{K}) \\ &+ \varepsilon^2 k_2 V_1(\mathcal{K}) \\ &+ \varepsilon^3 k_3 V_0(\mathcal{K}) \end{aligned}}$$

A Steiner walk-through, 3d

$$V' = V + \sum_i \epsilon A_i + \sum_j \epsilon^2 a\theta + \sum_l \epsilon^3 K_l$$

$$\boxed{\begin{array}{rcl} V_3(\mathcal{K}_\varepsilon) & = & k_0 V_3(\mathcal{K}) \\ & + & \varepsilon k_1 V_2(\mathcal{K}) \\ & + & \varepsilon^2 k_2 V_1(\mathcal{K}) \\ & + & \varepsilon^3 k_3 V_0(\mathcal{K}) \end{array}}$$

A Steiner walk-through, 3d

$$
\begin{aligned}
V' = V & \\
& + \sum_i \epsilon A_i \\
& + \sum_j \epsilon^2 a\theta \\
& + \sum_l \epsilon^3 K_l
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
V_3(\mathcal{K}_\varepsilon) &= k_0 V_3(\mathcal{K}) \\
&+ \varepsilon k_1 V_2(\mathcal{K}) \\
&+ \varepsilon^2 k_2 V_1(\mathcal{K}) \\
&+ \varepsilon^3 k_3 V_0(\mathcal{K})
\end{aligned}
}
$$

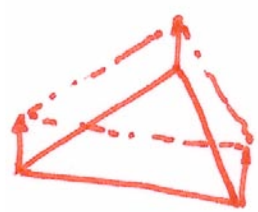
How to use?

Want to measure deformation?

- only very few measures need apply
  - volume, area, mean width, χ

Define curvatures of polyhedra?

- deal with convexity issue…



Normal cycle
Cohen-Steiner/Morvan