# Geometry Processing: A First Set of Applications

Peter Schröder

# In this Section

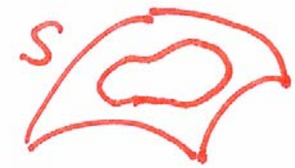
What characterizes shape?

- brief recall of classic notions
  - mean and Gaussian curvature
- how to express them in the discrete setting of meshes?
- putting them to work
  - smoothing
  - parameterization

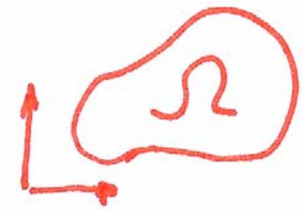
# Surfaces

## Basic setup

- parameterized surface

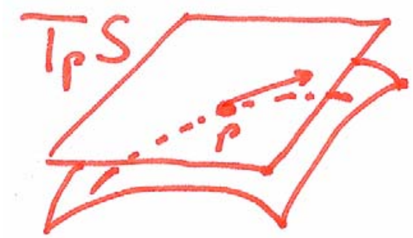



$$S : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^3$$
$$S(u,v) = (x(u,v), y(u,v), z(u,v))$$




- tangent vectors

$$c : I \to S \qquad c(0) = p \qquad \dot{c}(0) = \alpha$$




- tangent space $\qquad T_pS$

# Metric on Surface

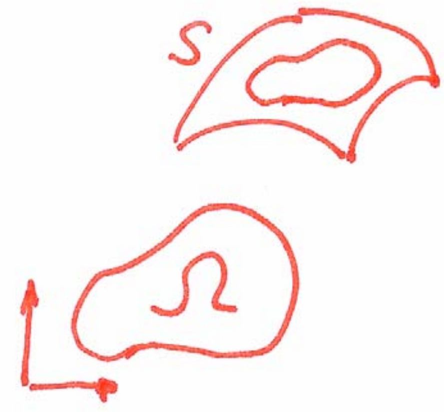
Measure stuff

- angle, length, area
  - all require an inner product $\langle v, w \rangle$
- we have:
  - Euclidean inner product in domain
- want to turn this into:
  - inner product on surface

# Lifting onto Surface

Use basis vectors

- du and dv in domain
- $S_{,u}$ and $S_{,v}$ on surface
  - record all their inner products

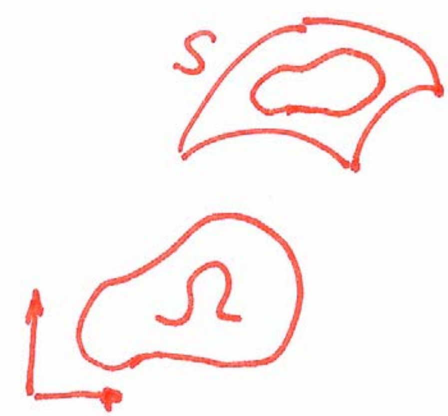



$$I_p = \begin{pmatrix} \langle S_{,u}(p), S_{,u}(p) \rangle & \langle S_{,u}(p), S_{,v}(p) \rangle \\ \langle S_{,v}(p), S_{,u}(p) \rangle & \langle S_{,v}(p), S_{,v}(p) \rangle \end{pmatrix}$$

# Lifting onto Surface

Use basis vectors

- du and dv in domain
- $S_{,u}$ and $S_{,v}$ on surface
  - record all their inner products



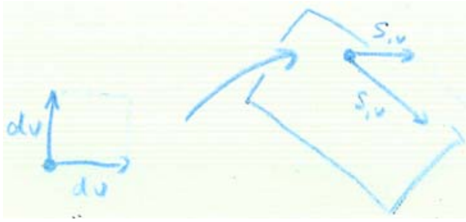

$$I_p = \begin{pmatrix} \Gamma & 0 \\ 0 & \gamma \end{pmatrix}$$

principal "stretches"

  - two invariants: trace & determinant
    - average length & area $\Gamma + \gamma$ $\Gamma\gamma$

# First Fundamental Form

## Measuring area



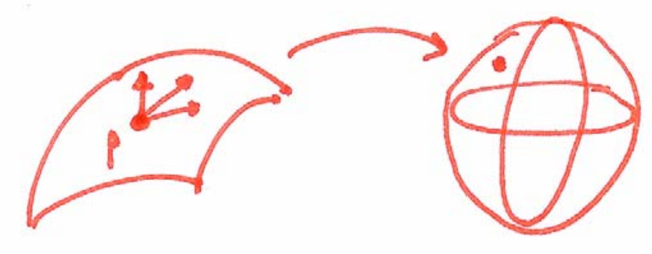
- areas in tangent space

$$\int\int_{\Omega} |S_{,u} \times S_{,v}| \, du \, dv = A(S) = \int_{S} 1 \, dA$$

  - no dependence on parameterization
- discrete setting... easy
  - sum areas of triangles

# Geometry of the Normal

## Gauss map

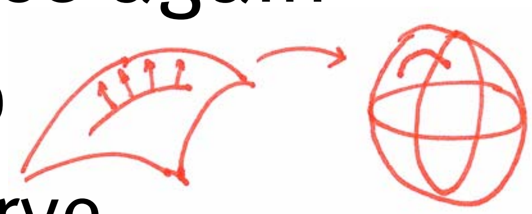


- normal at point

$$N(p) = \frac{S_{,u} \times S_{,v}}{|S_{,u} \times S_{,v}|}(p)$$

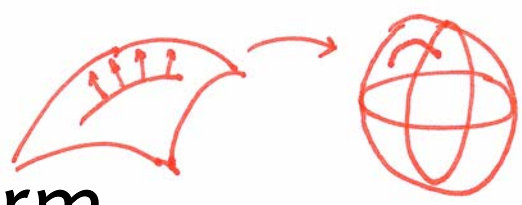
$$N : S \to \mathbb{S}^2$$

- consider curve in surface again
  - study its curvature at p
  - normal "tilts" along curve



# Shape Operator

Derivative of Gauss map



- second fundamental form

$$II_p(v) = \langle dN_p(v), v \rangle$$ ← self-adjoint

- local coordinates

$$II_p = -\begin{pmatrix} \langle N, S_{,uu} \rangle & \langle N, S_{,uv} \rangle \\ \langle N, S_{,vu} \rangle & \langle N, S_{,vv} \rangle \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

- unpleasant expression
  - invariants again...

# Invariants

## Gaussian and mean curvature

- determinant and trace only

$$\det dN_p = \kappa_1 \kappa_2 = K \qquad \text{intrinsic}$$

$$\operatorname{tr} dN_p = \kappa_1 + \kappa_2 = H \qquad \text{extrinsic}$$

- eigen values and (ortho) vectors

$$dN_p(e_1) = \kappa_1 e_1 \qquad dN_p(e_2) = \kappa_2 e_2$$

$$II_p|_{\mathbb{S} \subset T_pS} \quad \begin{cases} \max \to \kappa_1 \\ \min \to \kappa_2 \end{cases}$$

# Curvatures

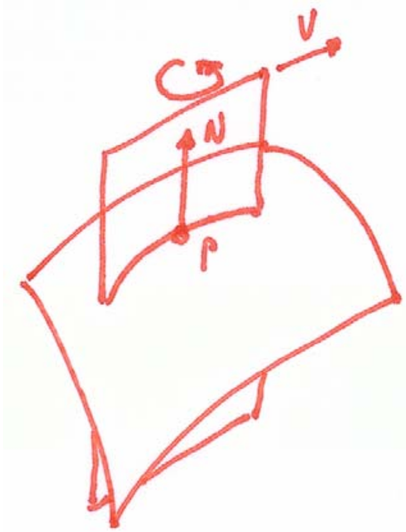
## Integral representations

- smooth setting

$$H_p/2 = \frac{1}{2\pi}\int_0^{2\pi} \kappa_n(\theta)\, d\theta$$

$$K_p = \lim_{A\to 0} \frac{A_G}{A}$$

- on a mesh?
  - hold on...




# Gaussian Curvature

On a mesh

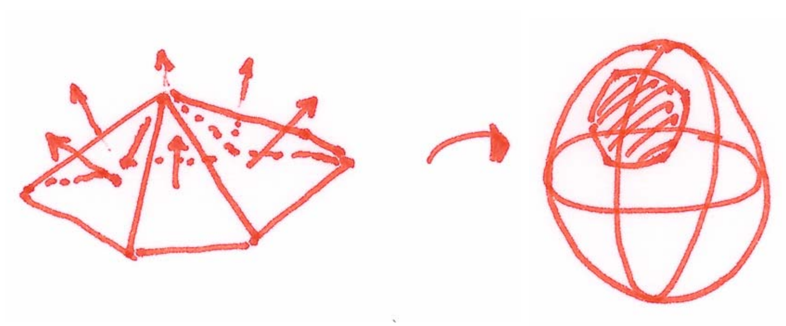
- can't take the limit...

$$K_p = \lim_{A \to 0} \frac{A_G}{A}$$

  - average does make sense

$$\int_A K_p \approx A K_p \approx A_G$$

only makes sense as an integral, NEVER pointwise



$$2\pi - \sum_{t_{pjk}} \alpha_{jk}$$

Discrete Gauss curvature at a vertex

# A Good Definition?

## Gaussian curvature over a surface

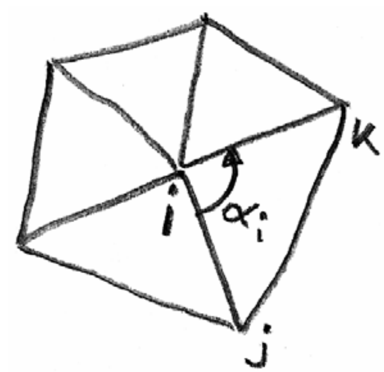
- Gauss-Bonnet

$2 - 2g$

closed, oriented

$$2\pi\chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA$$

- discrete




$$K_i = 2\pi - \sum_{t_{ijk}} \alpha_{jk}$$

$E$

$$\sum_i K_i = 2\pi(V - F/2) = 2\pi(F - 3F/2 + V) = 2\pi\chi$$

# Scalar Mean Curvature

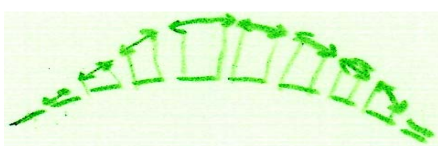
## Integral representation

- variation along a vector field

$$\partial_V \mathrm{Vol} = \int \langle V, N \rangle \, dA$$

$$H = \frac{\partial_V \mathrm{Area}}{\partial_V \mathrm{Vol}}$$

for constant V move out of integral



$$\partial_V \mathrm{Area} = \int \langle V, \mathbf{H} \rangle \, dA$$

$$\frac{|\int \mathbf{H} \, dA|}{|\int N \, dA|} \to H_p$$

# Boundary Integrals

## Vector area

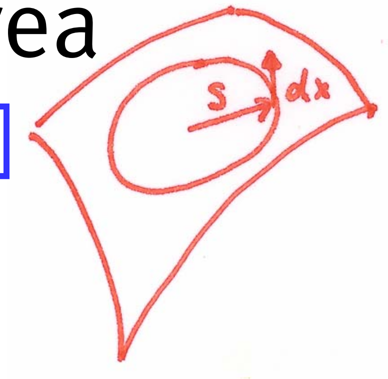
$$\frac{|\int \mathbf{H}\, dA|}{|\int N\, dA|}$$

- volume gradient: vector area

$$D \subset S \qquad \gamma = \partial D$$

another normal

$$\int_D N\, dA = 1/2 \oint_\gamma S \times dx = \mathbf{A}_\gamma$$

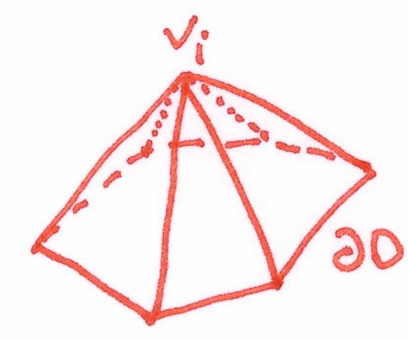



- discrete version

only makes sense as an integral, NEVER pointwise

$$3\mathbf{A}_i = 1/2 \sum p_j \times p_{j+1}$$

area weighted triangle normals



# Boundary Integrals

$$\frac{|\int \mathbf{H}\,dA|}{|\int N\,dA|}$$

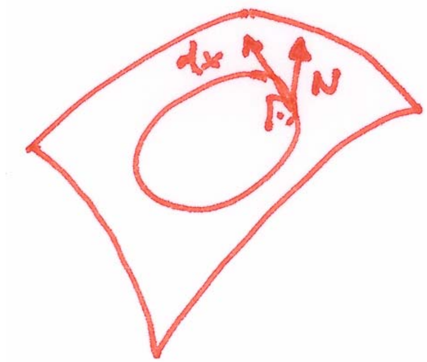
## Area gradient

- vector mean curvature

$$D \subset S \qquad \gamma = \partial D$$

another normal

$$\int_D \mathbf{H}\,dA = \oint_\gamma N \times dx$$

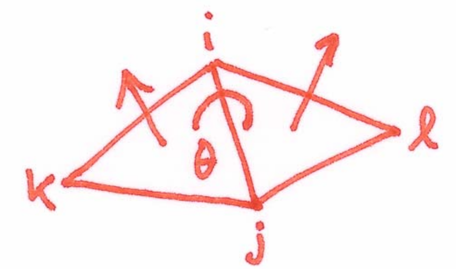



- discrete version

$$\mathbf{H}_e = e \times N_1 - e \times N_2$$

only makes sense as an integral, NEVER pointwise

$$|\mathbf{H}_e| = |e| 2 \sin\theta/2$$



# Boundary Integrals

$$\mathbf{H}_p = \lim_{A\to 0} \frac{\nabla A}{A}$$

## Area gradient

- vector mean curvature

$$D \subset S \qquad \gamma = \partial D$$

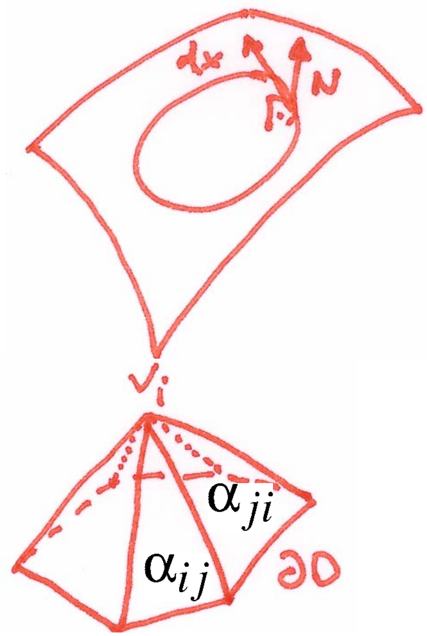
$$\int_D \mathbf{H}\, dA = \oint_\gamma N \times dx$$

another normal

- discrete version

$$2\mathbf{H}_i = \sum_j \mathbf{H}_{e_{ij}} = 2\nabla_i A$$
$$= \sum_j (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j)$$

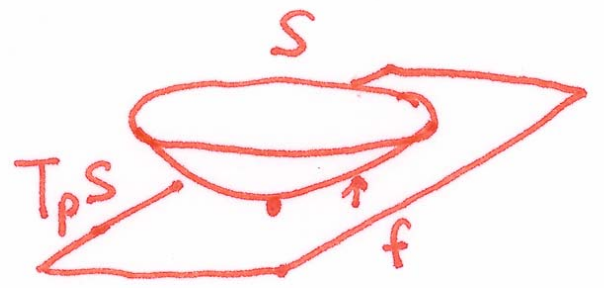
only makes sense as an integral, NEVER pointwise



# LAPLACE-(BELTRAMI)

Surface over tangent plane

- in eigen basis




$$H_p = \Delta f = (\frac{d^2}{du^2} + \frac{d^2}{dv^2}) f$$

principal curvature directions

- Laplace-Beltrami $\mathbf{H} = \Delta_S S$

Laplace on the surface

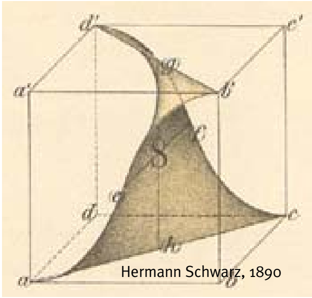
...of the surface

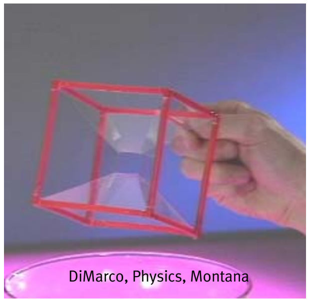
# Geometric Flow (Area)

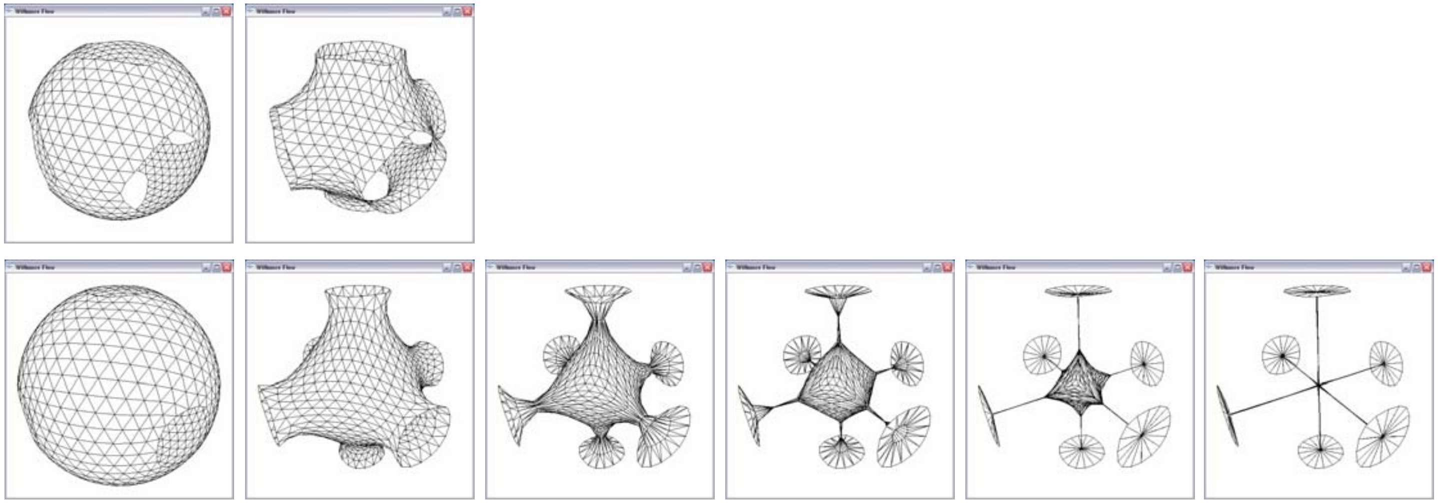
Minimize area energy

- minimal surface

$$E_A = \int_S 1\, dA \qquad S_t = -\nabla E_A$$



# Geometric Flow (Area)

## Minimize area energy

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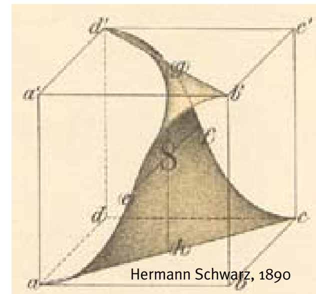
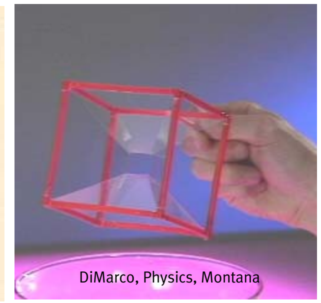
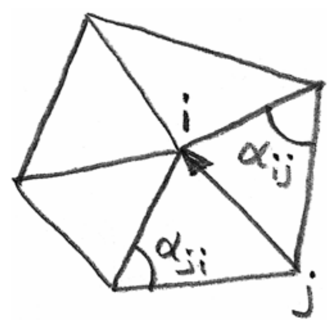
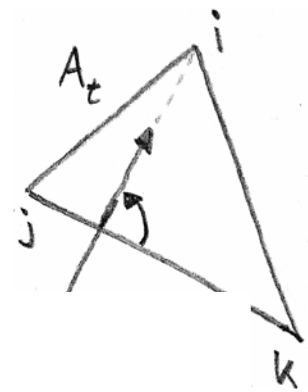

Hermann Schwarz, 1890


DiMarco, Physics, Montana

$$E_A = \int_S 1\, dA \qquad S_t = -\nabla E_A$$

$$2\partial_i A_{t_{ijk}} = R^{\pi/2}(p_k - p_j)$$




$$\begin{aligned} \partial_t p_i &= -\nabla_i E_A \\ &= -1/2 \sum_{e_{ij}} (\cot \alpha_{ij} + \cot \alpha_{ji})(p_i - p_j) \end{aligned}$$

# Mean Curvature Flow

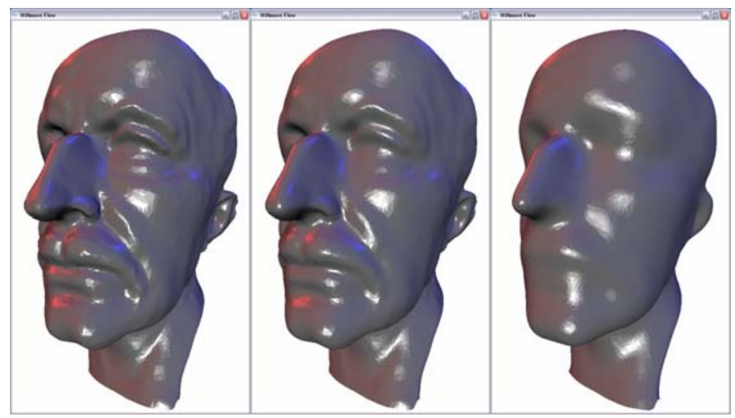
## Laplace-Beltrami

- Dirichlet energy

$$\min \int (\nabla u)^2 \rightsquigarrow \begin{array}{l} \Delta u = 0 \\ u|_{\partial\Omega} = u_0 \end{array}$$

- on surface

$$\mathbf{H} = \Delta_S S = \frac{\nabla A}{2A}$$



# Mean Curvature Flow

## Laplace-Beltrami

- Dirichlet energy

$$\min \int (\nabla u)^2 \rightsquigarrow \begin{array}{l} \Delta u = 0 \\ u|_{\partial\Omega} = u_0 \end{array}$$

- on surface

$$\begin{aligned} \partial_t p_i &= -\mathbf{H}_i \\ &= -1/4A_i \Sigma_{e_{ij}} (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j) \end{aligned}$$

# Parameterizations

## What is a parameterization?

- function from some region $\Omega \subset \mathbb{R}^2$ to the embedded surface $M \subset \mathbb{R}^3$

$$S(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$

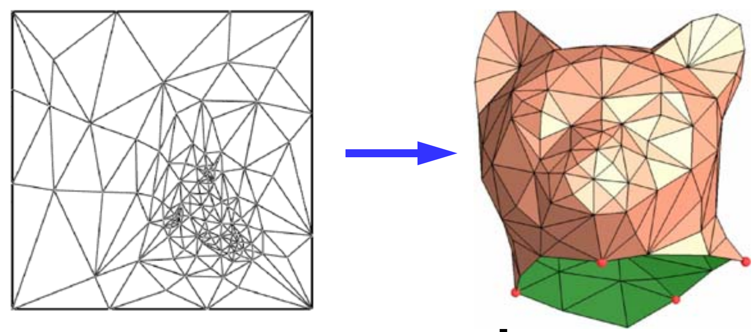


Image from Sander et al. 2002

- we go the other way around
- how to measure distortion?

# Measuring Distortion

## Dirichlet energy of a map

- discrete harmonic $\Delta_S \mathbf{u} = 0 \quad \mathbf{u}|_{\partial\Omega} = \mathbf{u}_0$

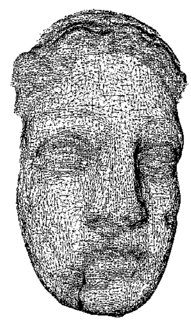
$$E_D(\mathbf{u}) = \int_S (\nabla_S \mathbf{u})^2 \, dA$$

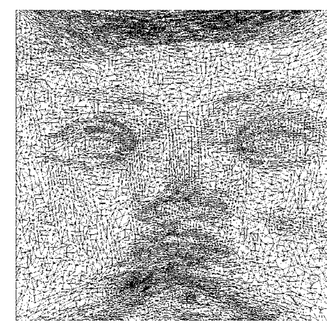
- minimizer is discrete harmonic

$$0 = \sum_j (\cot \alpha_{ij} + \cot \alpha_{ji})(\mathbf{u}_i - \mathbf{u}_j)$$

angles in mesh

texture coords.





- need to fix boundary

# Harmonic Map

## Properties of minimizer

- area minimization of triangle

$$\left\langle \frac{\partial s}{\partial u_1}, \frac{\partial s}{\partial u_2} \right\rangle = 0$$

$$\left| \frac{\partial s}{\partial u_1} \right| = \left| \frac{\partial s}{\partial u_2} \right|$$

conditions for u to be conformal

$$\begin{aligned} A &= \frac{1}{2} \int_T \left| \frac{\partial s}{\partial u_1} \times \frac{\partial s}{\partial u_2} \right| du \\ &\leq \frac{1}{2} \int_T \left| \frac{\partial s}{\partial u_1} \right| \left| \frac{\partial s}{\partial u_2} \right| du \\ &\leq \frac{1}{4} \int_T \left( \frac{\partial s}{\partial u_1} \right)^2 + \left( \frac{\partial s}{\partial u_2} \right)^2 du \end{aligned}$$

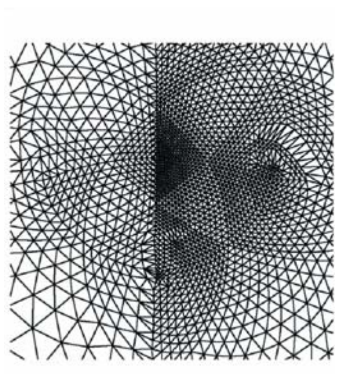
# Discrete Conformal

## Minimizer of conformal energy

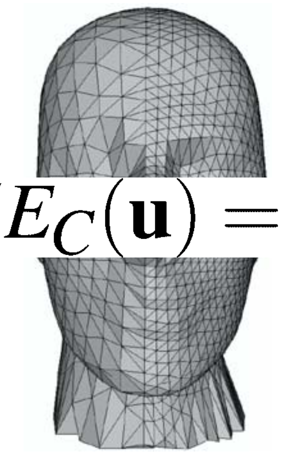
- $E_C(\mathbf{u}) = E_D(\mathbf{u}) - \text{Area}(\text{Range}(\mathbf{u}))$

Fixed boundary:
Dirichlet $\rightarrow$ constant

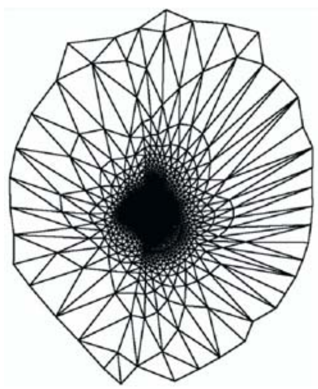
Free boundary:
Neumann $\rightarrow$ match gradients



Dirichlet



$$\nabla E_C(\mathbf{u}) = 0$$



Neumann

# Recap

Invariants as overarching theme

- shape does not depend on Euclidean motions
  - metric and curvatures
- smooth continuous notions to discrete notions
  - variational formulations
  - careful: generally only as averages

# Tools

Operators we have now

- volume gradient: notion of normal
- area gradient: notion of normal
  - also: mean curvature & LB
- smoothing, parameterization, editing (bi-Laplace-Beltrami)

Harmonic

Flow

Conformal

$$\Delta_S^2 \mathbf{S} = 0$$

G[1] boundaries to make smooth bump

# Down the Line

## Approach so far

- essentially linear: PL mesh...
- same equations can be derived with
  - DEC: discrete exterior calculus
  - abstract measure theory

## There is more

- some invariants are non-linear