Discrete Differential Geometry: An Applied Introduction

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with
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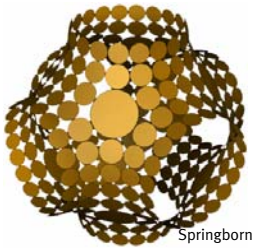
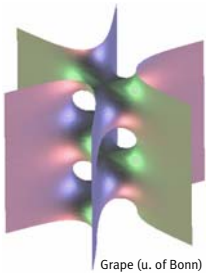
DDG Course SIGGRAPH 2005

Differential Geometry

Why do we care?

- geometry of surfaces


Springborn

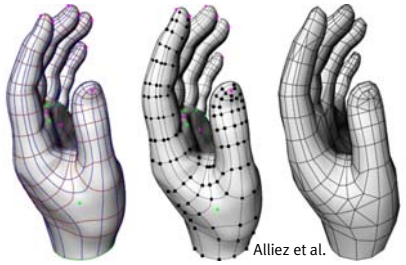

Grape (u. of Bonn)

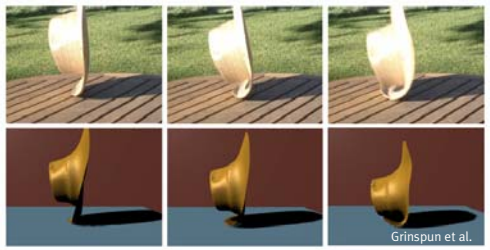
- mothertongue of physical theories

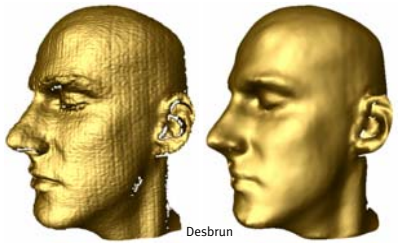
$$E = \int_S \alpha + \beta(H - H_0)^2 + \gamma K \, dA$$

- computation: simulation/processing


Elcott et al.


Alliez et al.

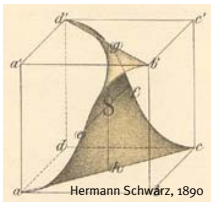

Grinspun et al.

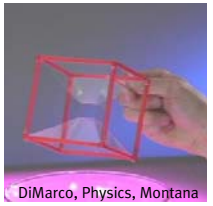

Desbrun

A Bit of History

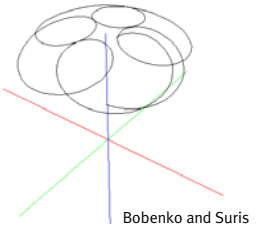
Geometry is the key!

- studied for centuries
  - Cartan, Poincaré, Lie, Hodge, de Rham, Gauss, Noether,...
- mostly differential geometry
  - differential and integral calculus


Hermann Schwarz, 1890


DiMarco, Physics, Montana

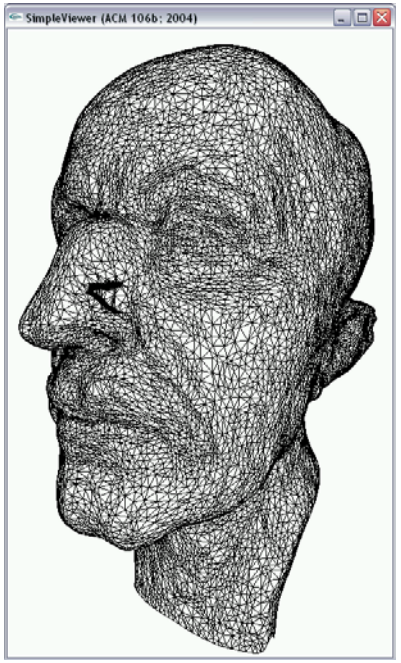
The study of invariants and symmetries


Bobenko and Suris

Getting Started

How to apply DiffGeo ideas?

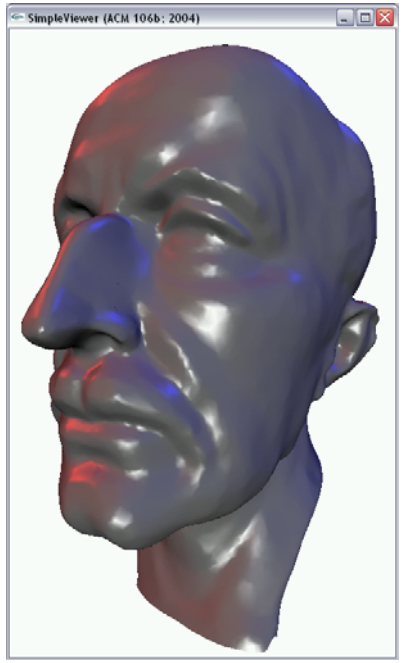
- surfaces as collections of samples
  - and topology (connectivity)



Getting Started

How to apply DiffGeo ideas?

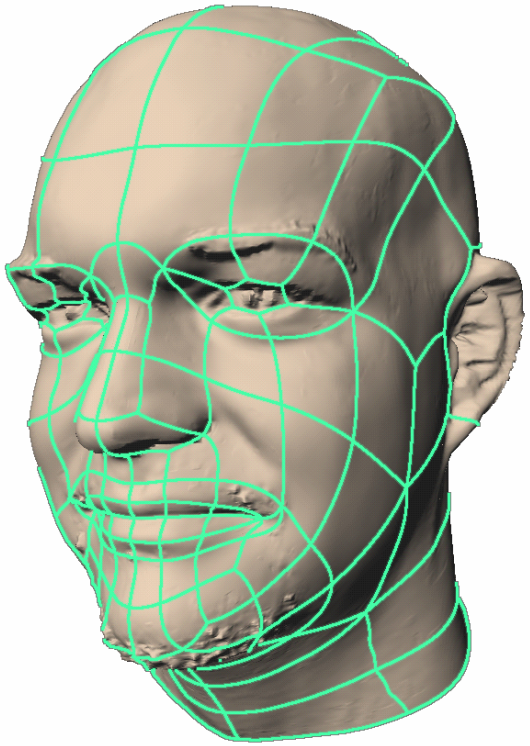
- surfaces as collections of samples
  - and topology (connectivity)
- apply continuous ideas
  - BUT: setting is discrete
- what is the right way?
  - discrete vs. discretized



Discretized

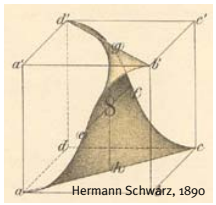
Build smooth manifold structure

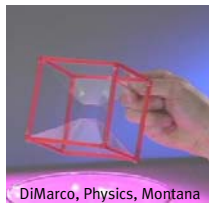
- collection of charts
  - mutually compatible on their overlaps
- form an atlas
- realize as smooth functions
  - differentiate away…



Discrete Geometry

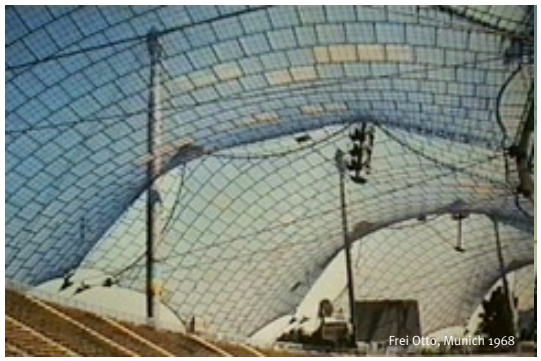
- Basic tool
  - differential geometry
    - metric, curvature, etc.
- Discrete realizations
  - "meshes"
    - computational geom.
    - graph theory


Hermann Schwarz, 1890


DiMarco, Physics, Montana


Uli Heller, 2002


Boy's Surface, Oberwolfach

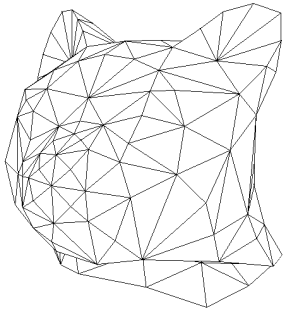

Frei Otto, Munich 1968

Discrete Diff.Geometry

Building from the ground up

- discrete geometry is the given
  - meshes: triangles, tets
  - more general: cell complex
- how to do calculus?
  - pick properties of import

$$\int_a^b f'(x)dx = f(b) - f(a)$$



What Matters?

- Structure preservation!
  - Symmetry groups
    - rigid bodies: Euclidean group
    - fluids: diffeomorphism group
    - conformal geometry: Möbius group
  - many more: symplectic invariants, Stokes' theorem, de Rham complex, etc. (pick your favorite)

Accuracy
Speed
Size

Propaganda

Themes for Today

What characterizes structure(s)?

- what is shape?
  - Euclidean invariance

- what is physics?
  - conservation/balance laws

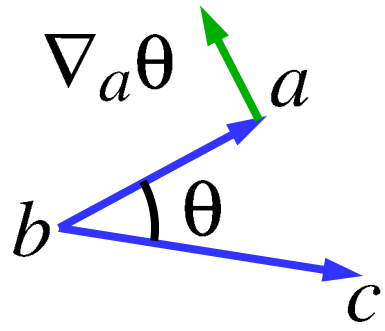
- what can we measure?
  - mass, area, curvature, flux, circulation

Themes for Today

- Invariant descriptions
  - quantities invariant under a set of transformations
    - symmetries give rise to momenta
- Intrinsic descriptions
  - quantities which do not depend on a coordinate frame

What it All Means

- Benefits
  - everything is geometric
    - often more straightforward
    - tons of indices verboten!

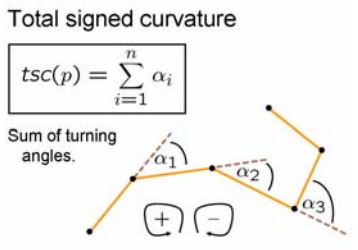


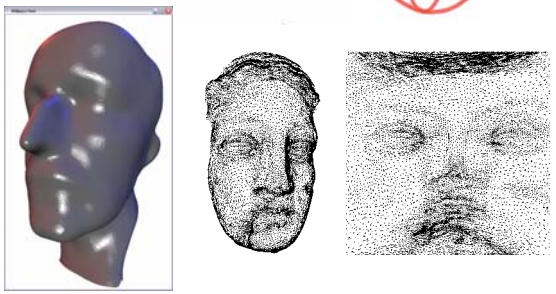
- The story is not finished…
  - still many open questions
    - in particular: numerical analysis

The Program for Today

Things we will cover

- warmup: curves
  - discrete analogues of cont. theorems
- surfaces: some basic operators
  - the discrete setting
  - putting them to work
    - denoising/smoothing, parameterization

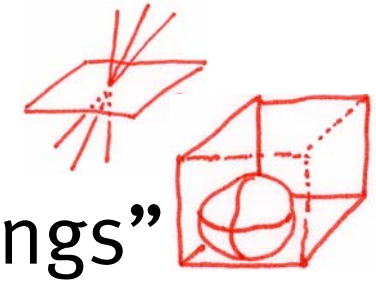


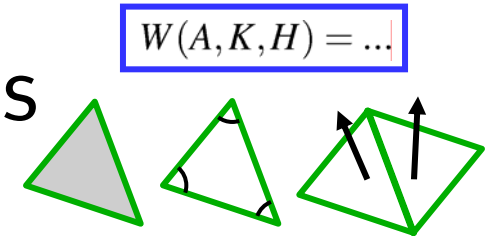


The Program for Today

Things we will cover

- what can we measure
  - invariant measures of “things”
    - curvature integrals without derivatives
- a first physics model
  - deformation of a shape
  - simulating discrete shells

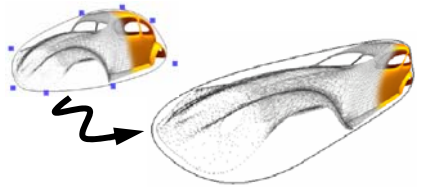




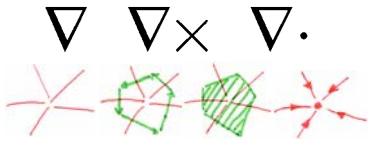
The Program for Today

Things we will cover

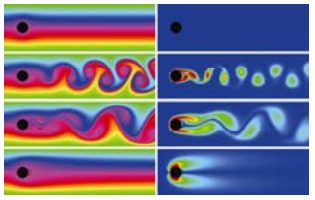
- interpolation on simplicial complexes, i.e., meshes



- discrete exterior calculus



- putting it to work: discrete fluids
  - structure preservation: vorticity
    - ensured by design!



The Program for Today

Things we will cover

- conformal geometry
  - conformal parameterizations
  - curvature energies
- how to make all those meshes
  - sampling a surface/volume
  - variational tet meshing