Discrete Differential Geometry: An Applied Introduction

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Differential Geometry

Why do we care?

- geometry of surfaces
- mothertongue of physical theories
- computation: simulation/processing

\[ E = \int_S \alpha + \beta (H - H_0)^2 + \gamma K \, dA \]
A Bit of History

Geometry is the key!
- studied for centuries
  - Cartan, Poincaré, Lie, Hodge, de Rham, Gauss, Noether,...
- mostly differential geometry
- differential and integral calculus

The study of invariants and symmetries
GETTING STARTED

How to apply DiffGeo ideas?
- surfaces as collections of samples
- and topology (connectivity)
How to apply DiffGeo ideas?

- surfaces as collections of samples
- and topology (connectivity)
- apply continuous ideas
- BUT: setting is discrete
- what is the right way?
- discrete vs. discretized
Discretized

Build smooth manifold structure

- collection of charts
  - mutually compatible on their overlaps
- form an atlas
- realize as smooth functions
  - differentiate away...
Discrete Geometry

Basic tool
- differential geometry
- metric, curvature, etc.

Discrete realizations
- “meshes”
- computational geom.
- graph theory

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Discrete Diff. Geometry

Building from the ground up

- Discrete geometry is the given
- Meshes: triangles, tets
- More general: cell complex

How to do calculus?

- Pick properties of import

\[ \int_{a}^{b} f'(x)dx = f(b) - f(a) \]
What Matters?

Structure preservation!
- Symmetry groups
  - rigid bodies: Euclidean group
  - fluids: diffeomorphism group
  - conformal geometry: Möbius group
- many more: symplectic invariants, Stokes’ theorem, de Rham complex, etc. (pick your favorite…)

Accuracy
Speed
Size
**Themes for Today**

What characterizes structure(s)?
- what is shape?
  - Euclidean invariance
- what is physics?
  - conservation/balance laws
- what can we measure?
  - mass, area, curvature, flux, circulation
Themes for Today

Invariant descriptions
- quantities invariant under a set of transformations
- symmetries give rise to momenta

Intrinsic descriptions
- quantities which do not depend on a coordinate frame
What it All Means

Benefits

- everything is geometric
- often more straightforward
- tons of indices verboten!

The story is not finished...

- still many open questions
  - in particular: numerical analysis
THE PROGRAM FOR TODAY

Things we will cover

- warmup: curves
- discrete analogues of cont. theorems
- surfaces: some basic operators
- the discrete setting
- putting them to work
- denoising/smoothing, parameterization
The Program for Today

Things we will cover

- what can we measure
  - invariant measures of “things”
  - curvature integrals without derivatives
- a first physics model
  - deformation of a shape
  - simulating discrete shells
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Things we will cover

- interpolation on simplicial complexes, i.e., meshes
- discrete exterior calculus
- putting it to work: discrete fluids
  - structure preservation: vorticity
  - ensured by design!

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Things we will cover
- conformal geometry
- conformal parameterizations
- curvature energies
- how to make all those meshes
- sampling a surface/volume
- variational tet meshing