Thin Shells & Curvature-Based Energy

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Thin shells and thin plates

Thin, flexible objects
Shells are naturally *curved*
Plates are naturally *flat*
Related work

Researchers in the graphics community:
- Terzopoulos, Bridson, Breen, etc.
  - ad-hoc models for cloth
- Bobenko & Suris, Pai
  - discrete models of elastic curves

[Choi and Ko]
Euler’s elastica

Early formulation of elastic curves

\[ E_{\text{bend}} = \int_0^l \kappa(s)^2 ds \]

Bernoulli began generalization to surfaces
Chladni’s vibrating plates

Plate vibrated by violin bow
Sand settles on nodal curves
Chladni’s vibrating plates

Plate of violin
Sand settles on nodal curves

Prize for explanation:
1kg of gold, 1808, 1811, 1815
Problem setup

What is the deformation energy?

undeformed body

\( \bar{x} \)

deformation

deformed body

\( x \)
Problem setup

What is the deformation energy?

undeformed config. is curved \Rightarrow \text{thin shell}

undeformed body

\bar{X}

defformation

defomed body

X
Problem setup

What is the deformation energy?

undeformed config. is flat \Rightarrow \text{thin plate}

undeformed body \quad \bar{x}

deformation

deformed body \quad x
Problem setup

Energy is a non-negative scalar function

Problem setup

Energy is a non-negative scalar function

real number, coordinate-frame invariant

each config. maps to a point on energy landscape

[T. L. Brown. Making truth]
Problem setup

Energy is a non-negative scalar function

[T. L. Brown. Making truth]
Problem setup

Internal forces push “downhill”

\[ f = -\nabla E \]

Plates

Germain

Poisson

Navier
Thin plate energy

Germain’s argument:
- bending energy must be a symmetric even function of principal curvatures
Thin plate energy

Germain’s argument:

• bending energy must be a symmetric even function of principal curvatures

\[
E^{\text{bend}} = f(\kappa_1, \kappa_2) = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 \, dA
\]

\[
= \int H^2 \, dA
\]
Thin plate energy

Poisson’s linearization

- assuming small displacements, approximate curvature by second derivatives

\[ E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 dA \]

\[ E^{bend}_{lin} = \int (\Delta f)^2 dA \]
Thin plate energy

Navier’s equation

• to find minimizer for linearized energy, solve a partial differential eqn (PDE)

\[ \Delta^2 f = 0 \]

\[ E_{lin}^{bend} = \int (\Delta f)^2 dA \]
Thin plate energy

Navier’s equation

- to find minimizer for linearized energy solver

\[ \Delta^2 f = 0 \]

\[ E_{\text{lin}}^{\text{bend}} = \int (\Delta f)^2 dA \]

\[ \partial_{uuuuu} f + 2\partial_{uuvv} f + \partial_{vvvv} f \]
Axiomatic approach

Energy should be:

- symmetric even func’n of principal curvatures
- extrinsic measure
- smooth w.r.t. change in shape
- invariant under rigid-body motion
- simple to compute
- easy to understand
What about masses and springs?

Diagonal springs don’t work for shells.

- undeformed configuration is *curved*
- incorrect energy minima
Axiomatic “discrete shells”

“Simplest” answer to desiderata

\[(H - H_0)^2\]

Derivation:
extrinsic change in \textit{shape} operator

\[\left[ \text{Tr}(\varphi^*S) - \text{Tr}(\bar{S}) \right]^2\]
Computing discrete shells

Elastic energy = \( \frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{||\bar{e}_i||}{\bar{h}_i} \)
Computing discrete shells

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Computing discrete shells

Elastic energy = \( \frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\|\bar{e}_i\|}{\bar{h}_i} \)

Gradient gives forces:

\[ f_k = K_B \sum_i \frac{\|\bar{e}_i\|}{\bar{h}_i} (\bar{\theta}_i - \theta_i) \nabla x_k \theta_i \]
Upgrade your cloth simulator

Have a cloth simulator handy?
- reuse all the existing code
- retrofit the bending term
- precompute undeformed quantities offline

\[ f_k = K_B \sum_i \frac{\|e_i\|}{h_i} (\bar{\theta}_i - \theta_i) \nabla x_k \theta_i \]
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$$ f_k = K_B \sum_i \frac{\|e_i\|}{h_i} (\bar{\theta}_i - \theta_i) \nabla x_k \theta_i $$
Modeling Paper

Paper sheet

- curled
- creased
- pinned
Are we done?

Discrete shells is nice and simple. What’s next?
Thin shell theory

Kirchhoff  Love  Karman  Koiter
Shell geometry

Shell representation: middle surface + normal offset
Stored energy

Step 1: strain
  • deformation of small volume

Step 2: energy density
  • compute work
  • constitutive model

Step 3: integrate
  • over shell thickness
  • over middle surface
Assumptions

Thin shell

• thickness much less than radius of curvature
Assumptions

Kirchhoff-Love

• normal lines deform to normal lines
Assumptions

Planar stress

- neglect stress in normal direction

[Diagram showing undeformed and deformed states with restoring forces lying on tangent plane]
Assumptions

Normal inextensibility

- distance preserved along normal lines
Assumptions

Small strains

• all strains are small
  • strains in glass: O(0.0001)
  • strains in paper: O(0.01)
• deflections may be large
Shell geometry

Locally description

- consider small neighborhood
- global parameterization *not* required
Deformation mapping

Maps material point on tangent plane from undeformed to deformed config.

$$\Psi$$

$$\bar{x} \rightarrow x = \Psi(\bar{x})$$
Deformation gradient

Maps tangent vector from undeformed to deformed config.

\[ t = \nabla \psi \bar{t} \]
Strain

Change in squared length between two nearby points

\[ l^2 - \bar{l}^2 \]
Strain

Change in squared length of tangent vector

\[ t = \nabla \psi \bar{t} \]
Strain

Change in squared length of tangent vector

\[ |t|^2 - |\bar{t}|^2 = t^T t - \bar{t}^T \bar{t} \]
Strain

Change in squared length of tangent vector

$$|t|^2 - |	ilde{t}|^2 = t^T t - \tilde{t}^T \tilde{t}$$

$$t = \nabla \psi \tilde{t}$$

$$t^T = \tilde{t}^T (\nabla \psi)^T$$
Strain

Change in squared length of tangent vector

$$\bar{t}^T (\nabla \psi)^T (\nabla \psi) \bar{t} - \bar{t}^T \bar{t}$$
Strain

Change in squared length of tangent vector

$$\bar{t}^T \left( \nabla \psi \right)^T \left( \nabla \psi \right) \bar{t} - \bar{t}^T \bar{t}$$

$$\bar{t}^T \left( \left( \nabla \psi \right)^T \left( \nabla \psi \right) - I \right) \bar{t}$$
Strain

Change in squared length of tangent vector

$$\bar{t}^T (\nabla \psi)^T (\nabla \psi) \bar{t} - \bar{t}^T \bar{t}$$

$$\bar{t}^T ((\nabla \psi)^T (\nabla \psi) - I) \bar{t}$$
Strain

Change in squared length of tangent vector

\[ E_m \]

\[ (\nabla \psi)^T (\nabla \psi) - I \]
Strain

Change in squared length of tangent vector

\[ \overline{t}^T E_m \overline{t} \]
Thought experiment

begin at rest,

\[ r_1 = (I + E_m)r_0 \]

...stretch the shell,

\[ r_2 = (I + zE_c)r_1 \approx (I + E_m + zE_c) \]

...then bend the shell

Membrane + bending strain

\[ E(z) = E_m + z\Delta\Lambda = E_m + zE_c \]
Surface Energy Density

Energy formulation - bending term

\[ \frac{Y h^3}{24(1 - \nu^2)} \left( (1 - \nu)\text{Tr}(E_c^2) + \nu(\text{Tr}E_c)^2 \right) \]

- geometric interpretation

\[ (1 + \nu) \Delta H^2 + (1 - \nu)(\Delta A^2 + 4A\tilde{A}\sin^2 \beta) \]

  - change in mean curvature
  - change in curvature direction
Planar strain

Consider a single deformed triangle…
Planar strain

Function only of change in edge lengths!
• (thank you Hadwiger!)
Planar strain

Function only of $\alpha$

• (thank you Hao)

\[
\begin{align*}
    s_1 &= a_1^2 - \bar{a}_1^2 \\
    s_2 &= a_2^2 - \bar{a}_2^2 \\
    s_3 &= a_3^2 - \bar{a}_3^2
\end{align*}
\]
Planar strain

\[ s_1 = a_{11}^2 - \overline{a}_{11}^2 \]
\[ s_2 = a_{22}^2 - \overline{a}_{22}^2 \]
\[ s_3 = a_{33}^2 - \overline{a}_{33}^2 \]

\[ E_m = \frac{1}{8A^2} \sum_{i=1}^{3} s_i \left( t_j \otimes t_k + t_k \otimes t_j \right) \]
Discrete bending strain

\[ E_b = \frac{1}{2A} \sum_{i=1}^{3} (\theta - \bar{\theta}) \frac{t_i \otimes t_i}{a_i} \]
Material Failure

- Stress
- Strain
- Critical Stress
- Fracture
- Plastic-Elastic Regime
- Elastic Regime

Graph showing the relationship between stress and strain, indicating failure at the critical stress point.
Material Failure

Principal strains

- compute maximal strain

\[ E_m \pm hE_c \]

- eigenvalue > threshold
  - split against eigenvector
Results: plastic deformation

Falling Tube
Results: plastic deformation

Falling Tube
Results: Fracture
Results: Fracture
Results: Fracture
Results: Fracture
Results: Fracture

Lightbulbs
Results: Fracture

Lightbulbs
What about convergence?

With sufficient refinement, does the discrete energy agree with the continuous energy?

Work in progress…
Convergence... why it matters

Maybe small systematic error is ok?
  • mesh independence for fine meshes
  • adaptive refinement
    • requires error criteria
    • .: limit must be well-defined
  • for simulation
    • can use physical parameters
  • for variational modeling
    • canonical surface definition
Discrete curvature measures

Basic building block \( H_e = n_e |e| f\left(\frac{\theta}{2}\right) \)

- total curvature normal
Discrete curvature measures

Basic building block \( H_e = n_e |e| f\left(\frac{\theta}{2}\right) \)
- total curvature normal

Variations of total mean curvature
- per edge    per triangle    per vertex
Discrete curvature measures

Basic building block \( H_e = n_e |e| f\left(\frac{\theta}{2}\right) \)

- total curvature normal

Variations of total mean curvature
- per edge  per triangle  per vertex

\[ \theta_e |e| \]

discrete shells, hinge energy
Discrete curvature measures

Basic building block \( H_e = n_e |e| f\left(\frac{\theta}{2}\right) \)
- total curvature normal

Variations of total mean curvature
- per edge per triangle per vertex

\[ \theta_e |e| \quad \frac{1}{2} \left( \sum_{e} \theta_e |e| \right) \]
- discrete shells, hinge energy
- triangle-averaged
Discrete curvature measures

Basic building block \( H_e = n_e |e| f\left(\frac{\theta}{2}\right) \)
- total curvature normal

Variations of total mean curvature
- per edge
- per triangle
- per vertex

\[ \frac{1}{2} \left( \sum_e \theta_e |e| \right) \]
- triangle-averaged

\[ \frac{1}{2} \left| \sum_e H_e \right| \]
- vertex-averaged

\[ \theta_e |e| \]
- discrete shells, hinge energy
Discrete curvature measures

Basic building block \( H_e = n_e \theta |e| f\left(\frac{\theta}{2}\right) \)
- total curvature normal

Variations of total mean curvature
- per edge \( \frac{1}{2} \left( \sum_e \theta_e |e| \right) \) (triangle-averaged)
- per vertex \( \frac{1}{2} \left| \sum_e H_e \right| \) (vertex-averaged)
- discrete shells, hinge energy \( \theta_e |e| \)
- per triangle \( \frac{1}{2} \left| \sum_e H_e \right| \) (vertex-averaged)
- per vertex \( \frac{1}{2} \left( \cot \alpha_e + \cot \beta_e \right) |e| \)
Test problem

4-8 patch and 1:2 aspect ratio stripe
flat plate, boundary prescribed by quadratic polynomial
interior free to assume minimal-energy
Test problem

4-8 patch and 1:2 aspect ratio stripe
flat plate, boundary prescribed by quadratic polynomial
interior free to assume minimal-energy
Shading and highlight maps

- discrete shell
- triangle averaged
- quadratic fit

4-8 patch

1:2 aspect ratio
Reflection maps

discrete shell
triangle averaged
quadratic fit
The next step

Need better handle on convergence and mesh-independence

Plus, keep the good stuff:

• geometric invariance:
  • rigid transforms and scaling
• efficiency
  • low FLOPS for gradient and Hessian
Convergence: necessary conditions

Claim

Two necessary conditions must be met so that for sufficiently fine meshes, simulation results will be:

• independent of chosen mesh
• in agreement with continuous theory.
Convergence condition 1

“The linearized energy of any sampled quadratic polynomial should be reproduced exactly.”

• choose any quadratic polynomial surface
• sample the surface (even with a coarse mesh)
• measure the linearized energy
• must exactly match the continuous energy
Convergence condition 2

“If the boundary conditions are sampled from a quadratic polynomial, the minimizer of the discrete energy should be exactly the sampled polynomial.”

- choose any quadratic polynomial surface
- constrain mesh boundary to samples of poly. surface
- let mesh interior relax to minimum energy
- mesh interior must lie exactly on poly. surface
Convergence for general meshes

- splines/subdiv. surfaces:
  no quadratic polynomial reproduction near extr. vertices
- discrete shells:
  no quadratic polynomial reproduction
- triangle quadratic fit:
  reproduces quadratic polynomials, but there can be lower energy states
- triangle averaged:
  no quadratic polynomial reproduction
- vertex averaged:
  no quadratic polynomial reproduction
- Compare: finite elements only have aspect ratio restrictions
Minimal d.o.f. discretizations

Question:  
*can we get away with only 6 d.o.f. per energy term?*

Answer:  
*yes, but we must permit d.o.f.s on edges.*
Midedge normal

4-8 patch

1:2 aspect ratio
Reflection maps

discrete shell  triangle averaged  quadratic fit  midedge normal
Mathematics

T. J. Willmore’s surfaces
Mathematics

T. J. Willmore’s surfaces

\[ \frac{1}{4} \int (\kappa_1 - \kappa_2)^2 \, dA = \int (H^2 - K) \, dA \]
Mathematics

T. J. Willmore’s surfaces

\[ \frac{1}{4} \int (\kappa_1 - \kappa_2)^2 dA = \int (H^2 - K) dA \]

\[ \int H^2 dA = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 dA \]
Physics of membranes

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Physics of membranes

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\[ \int \left( H - H_0 \right)^2 dA \]
Engineering

Civil, mechanical, aeronautical design
Geometric modeling

Surface fairing and reconstruction
Geometric modeling

Variational modeling

M. Botsch L. Kobbelt
Physically-based animation

Shells: elasticity, plasticity, fracture
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