Applications of DEC:
Fluid Mechanics and Meshing
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Overview
Putting DEC to good use

Fluids, fluids, fluids
- geometric interpretation of classical models
- discrete geometric interpretation
  - new geometry-based integration technique

Quadrangle Meshing
- discrete notion of harmonicity
- practical method to directly create quads

Part I
Computational Fluids with DEC
based on work with Sharif Elcott, Yiying Tong, Eva Kanso, Peter Schröder

Fluid Models (I)
Euler Equations
\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + f - \nabla \cdot u = 0
\]

- inviscid fluids (not viscous)
- incompressible
- non-linear PDE, with linear constraint

Fluid Models (II)
Navier-Stokes Equations
\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + f - \nu \Delta u\]

- only change: viscosity
  - coefficient \(\nu\)
- loss of total energy during motion
“Geometry” of Fluids? \[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p \]

Euler equations seem clear
- advection + div-free projection ad infinitum
- Stam’s Stable Fluids do this wonderfully well
- numerous follow-up work (Fedkiw et al.)
- but what does it mean, geometrically?
- “total energy” is rather unintuitive
- is there a notion of momentum preservation?

Yes
- but of course, we need to massage the PDE
- so as to reveal the geometric structure

Geometry Revealed

So we know:
Integral of vorticity constant on advected sheet

Additionally, \( \omega \) defines \( u \)
- if we ignore complex topology for a moment
- \( u = \nabla \times (\Delta^{-1} \omega) \) because \( u \) is divergence free!

Vorticity is the only real variable here and Kelvin’s is a defining property (Navier-Stokes: loss along the way)

Towards a Proper Discretization

Domain discretization = simplicial complex
- fluxes through faces for velocity
- intrinsic (coordinate-free) and eulerian
  - reminiscent of staggered grids...
- net flux for divergence
  - what comes in...must come out
- flux spin for vorticity
  - Torque created on a “paddle wheel”
- valid for any grid...

Enter Discrete Exterior Calculus

Need for proper link btw flux, vorticity, div
- hopefully matching differential counterparts
- to create a discrete differential structure
  - i.e., structure-preserving discretization
- Fortunately, that’s DEC
  - we know how to do all that, right??
  - flux = 2-form
  - div - exterior derivative of flux
  - curl = \( \star \nabla \star \) of flux

Divergence Operator

Simply \( d \) of 2-form
- summing face values of tets
- returning values in tets
**Curl Operator**

Curl requires going to the dual
- from faces to dual edges first
- then $d$ (sum of dual edge values)
- then back onto primal edges

- point-based scalar field
- cell-based scalar field
- face-based vector field
- edge-based vector field
- cell-based vector field
- volume scalar field

**Laplacian Operator**

For Navier-Stokes, Laplacian needed
- from faces to faces

- point-based scalar field
- cell-based scalar field
- face-based vector field
- edge-based vector field
- cell-based vector field

Integrating Equations of Motion

- We have all the computational set-up
- But how do we integrate the motion?
- Through preserving important structures?
  - Circulation/vorticity preservation
  - Crucial for visual impact
    - volutes in smoke
    - vortices in liquids

Discrete Kelvin’s Theorem

Guarantees circulation preservation...
- for any discrete loop!
- big loop = union of small ones
- ... even on curved spaces

- Difference with Stable Fluids?
  - trace back integrals, not point values

Results

- New method
  - exact discrete vorticity preservation
  - arbitrary simplicial meshes
  - see also [Feldman et al. ’05, Klingner et al. ’06]
  - everything is intrinsic
  - basic operators very simple (super parse)
  - great flows for small meshes!
  - computationally efficient even on coarse mesh
  - no need for millions of vortex particles
Channel

Smoking Bunny

Merging Vortices

Movie

Part II

Quad Meshing with DEC

Based on work with Yijing Tong, Pierre Alliez, David Cohen-Steiner

Quadrangulations

Needed in CAGD, Reverse Engineering

- Ubiquitous (tensor-product nature)
  - Modeling anisotropy/symmetries
  - FEM, texture atlas
- But global topology constraints...

A Variety of Requirements:

- Isotropy vs anisotropy
- Orthogonality, Alignment
- Regularity, Sizing
Quad Meshes: Reverse Engineering

For a local patch of quadrangulation

- Induce natural (u,v) parametrization
- Edges: integer-valued isocurves of u/v
- “Nice” mesh: square mesh in certain metric

\[ \langle \nabla u, \nabla v \rangle = 0 \] \[ \langle \nabla u, \nabla u \rangle - \langle \nabla v, \nabla v \rangle \]

- Cauchy–Riemann equations
- Using language of differential form, this is

\[ du \approx * dv \]

Thus, \( u \) and \( v \) are both harmonic (Laplacian=0)

\( du \) and \( dv \) too! Cool, DEC seems perfect for that

Methods for Quadrangulations

Among many:

- Clustering/Morse [Boier-Martin et al. 03, Carr et al. 06]
- Curvature lines [Alliez et al. 03, Marinov/Kobbelt 05]
- Isocontours [Dong et al. 04]
  - Two continuous potentials
  - (Much) more robust than streamlines
- Periodic global param (PGP) [Ray et al. 06]
  - Pbs: PGP non-linear + no real control

What about using discrete forms?

- Global conformal param [Gu/Yau 03]

Problems, Problems...

Solving for two continuous potentials (u,v)

- With gradients fields satisfying CR eqs
- Alas, singularities unavoidable

  - Either poles
  - Or line singularity

  - T-junctions...

Discontinuous Potentials

“Tweaked” Laplacian

- Continuity of 1-form induces:

\[ du^- = du^+ \]

\[ u^- = u^+ + V_i \]

Similarly, \( v^- = v^+ + V_i \)

\[ \sum_{j \in \mathcal{N}(i)} (\cot \alpha_{ij} + \cot \beta_{ij}) \left( \frac{u_i - u_j}{v_i} \right) - \sum_{j \in \mathcal{N}(i)} (\cot \alpha_{ij} + \cot \beta_{ij}) \left( \frac{u_j}{v_i} \right) \]

Generate smooth fields modulo the jump!
Simple Example of Tweaked $\Delta$

Once isolines of $u$ and $v$ are extracted:

$u$ (discontinuous)

$v$ (continuous)

Possible Discontinuities

Only three different cases:

- only way to guarantee pure quads
- in the 3 cases, just a tweak of the Laplacian
  - still only a linear system to solve!

Examples

Example: Pure-Quad Bunny

Advantages of DEC

Foundations of discrete forms powerful

- good grasp on vector field singularities
- control of "irregular valences"
- link with "cone singularities" [Kharevych et al. 06]
- still just a linear system to solve
- no need for well-tailored cuts
- provides parameterization too!
Take-Home Message

Don’t Arbitrily Discretize!
- discretize geometric structures
  - PDEs often hide these structures
- uncover the nature of the variables involved
  - usually, natural locations on mesh
- turn the crank with some DEC tools...

Next
- circles may be the right discrete geometry!
  - conformal geometry