Big Picture

Deriving a whole *Discrete Calculus*
- you need first a *discrete domain*
  - will induce the notion of *chains*
    - discrete representation of geometry
- then *discrete “differential” operators*
  - applied to our discrete geometric set-up
  - defined through *cochains* (discrete forms)

*Scared?*
- Don’t be: just numbers on mesh elmts!

Part I

Discrete Geometric Setup

Discrete Setup

Starting with a *discrete domain*
- can be thought of as “approximation”
  - cell decomposition of smooth manifold

Nice simplicial mesh
- vertex, edge, triangles, ...
- 2D domain

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- curved 2D domain
- 3D domain
- or even 4D and more!

Quick Refresher Course
Notion of k-simplex
- non-degenerate convex hull of (k+1) vertices
  - 0-simplex
  - 1-simplex
  - 2-simplex
  - 3-simplex
- boundary of simplex?
  - $\partial \sigma = \sum_{j} \beta_j \sigma_j$
- simplicial complex
  - your typical triangle mesh/tet mesh
  - two k-simplices only intersect thru a common (k-1) face

Discrete Subdomains
How to define geometric subsets?
- again, the only “medium” we have is the mesh
  - simple example:

Notion of Chains
Allowing linear combination of simplices
- assign coefficients to simplices
  - not just 0 or 1
  - it’s called a chain
  - we’re summing up like the sum of an edge
Definition:
- k-chain = one value per k-simplex
  - think “column vector”
  - ... or simply an array of values in CS terms

Chains and Boundaries
Boundary of a chain is a chain:
$\partial \kappa = \sum_{\sigma_j \in K \setminus \kappa} \sigma_j$  
- chains allow
  - “sub-simplex” accuracy
  - anti-aliased version...
- chains extension of barycentric coordinates!
- notion of homology
Notion of Dual Complex

Associate to each k-simplex a (n-k)-cell
- connectivity of mesh induces another mesh
- "Voronoi" diagram

Chains are defined as well on dual mesh!

Intrinsic Calculus on Meshes

Ready to bootstrap a discrete calculus
- using only values on simplices
  - "measurements" in the domain
- preview:
  - pre bond scalar field \( \frac{\nabla}{d} \)
  - edge bond vector field \( \frac{\nabla \times}{d} \)
  - face bond vector field \( \frac{\nabla}{d} \)
  - cell bond scalar field
- deep roots in mathematics
  - exterior calculus, algebraic topology
- but very simple to implement and use

Forms You Know For Sure

Digital Images: 2-forms
- incident flux on sensors (W/m²)
- "hmm, looks like a chain, no?" (almost)

Magnetic Field \( \mathbf{B} \): 2-form
- only measurement possible: flux!
- any physical flux is a 2-form too

Electrical Force \( \mathbf{E} \): 1-form
- any physical circulation is a 1-form too

Discrete Differential Quantities

Mentioned repeatedly in the talks before...
- they "live" at special places, as distributions
  - Gaussian curvature at vertices ONLY
  - mean curvature at edges ONLY
- they can be handled through integration
  - integration calls for k-forms (antisymmetric tensors)
    - objects that begin to be integrated (ex: \( \int f(x) dx \))
  - k-forms are evaluated on kD set
    - 0-form is evaluated at a point,
      1-form at a curve, etc...
    - what you can "measure"

Exterior Calculus of Forms

Foundation of calculus on smooth manifolds
- Historically, purpose was to extend div/curl/grad
  - Poincaré, Cartan, Lie, ...
- Basis of differential and integral computations
  - highlights topological and geometrical structures
  - modern diff. geometry, Hodge decomposition, ...
- A hierarchy of basic operators are defined:
  - \( d, \wedge, \delta, \partial, \nabla \cdot, \nabla \times \)
- See [Abraham, Marsden, Ratiu], ch. 6-7

Turning our Mesh into a Computational Structure
**Discrete Forms?**

**Idea: Sampling Forms on Each Simplex**
- Extends the idea of point-sampling of functions
- "Sample" (i.e., integrate) a k-form on k-cells
- The rest is defined by linearity

\[ \int_{\sigma_j} \omega = \sum_{j} \int_{\sigma_j} \omega \]

- Ex: if we know the flux on each edge, flux over the boundary of triangle is just the sum of the fluxes on the edges.

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**Discrete Forms As Cochains**

Discrete k-form - values on each kD set
- Primal forms on simplices, dual forms on cells
- Not a chain! Think "row vector" this time
  - In CS terms: k-form - array of values too

These discrete forms are cochains
- In math terms: chains pair with cochains (natural pairing - integration)
- If chain \( c \) valued \( c[i] \) on \( \sigma_j \), form \( \omega \) valued \( \omega[i] \) on \( \sigma_j \)

\[ \langle \omega, c \rangle = \int_{c} \omega = \int \omega = \sum_{j} c[i] \int_{\sigma_j} \omega = \sum_{j} \int_{\sigma_j} \omega[i] \]

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**Notion of Exterior Derivative**

Stokes/Green/... theorem:
\[ \int d\omega = \int \omega \]

- \( d \) and \( \partial \) are dual
- Implementation?
  - As simple as an incidence matrix!
  - Ex: \( d(1\text{-form}) \) - incidence matrix of edges & faces

Bean counting: array\([|F|\times|E|]\) x array\(|E| = array\(|F|\]

**Exterior Derivative**

Let’s try

- No “metric” needed! (no size measurement)
- Try \( d \), then \( d \) again on an arbitrary form...
  - Zero, why?
  - Because \( \partial \circ \partial = 0 \)
  - Good: \( \text{div}(\text{curl}) = \text{curl}(\text{grad}) = 0 \) automatically

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**Hodge Star**

Take forms to dual complex (and vice-versa)
- Switch values btw primal/dual
  \[ \ast : \Omega^p \rightarrow \Omega^{n-p} \]
- “Diagonal” hodge star
  \[ \frac{1}{\text{vol}\, \sigma^p} \int_{\sigma^p} \ast \omega^p = \frac{1}{\text{vol}\, \sigma^p} \int_{\sigma^p} \ast \omega^p \]
- Again, a simple (diagonal) matrix
- Now the metric enters...
- Hodge star defines accuracy

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**Discrete deRham Complex**

Discrete calculus through linear algebra:
- Simple exercise in matrix assembly
- All made out of two trivial operations:
  - Summing values on simplices \( (d/\partial) \)
  - Scaling values based on local sizes \( (*) \)
Interpolating Discrete Forms

Whitney basis functions to interpolate forms

- 0-forms (functions)
  - linear “hat” functions
- 1-forms (edge elements)
  - Whitney forms: \( \phi_{ij} = \phi_i \varphi_j - \phi_j \varphi_i \)
  - basis of 1-forms since \( \delta \phi_{ij} = 0 \) on edges \( i,j \)
- 2-forms: face elmts
- 3-forms: constant per tet

Higher order bases? See [Ke Wang et al. 2006]

What We’ll Be Able To Do

Interpolating Discrete Forms

Other Related Research Areas

Geometric Algebra
Mimetic Differencing

- same spirit, less geometry

Finite {Element|Volume|Differences}
Discrete Mechanics

- what about time discretization now?
- principle of least-action is crucial
  - motion is a geodesic if we use action as “metric”
- Last talk of the day

Take-Home Message

Geometric Approach to Computations

- discrete setup acknowledged from the get-go
- choice of proper habitat for quantities
- whole calculus built using only:
  - boundary of mesh elements
  - scaling by local measurements
- preserving structural identities
  - they are not just abstract concepts
  - they represent defining symmetries

Join the Discrete World

Wanna know more about DEC?

- Chapter in course notes
  - “Discrete Differential Forms” by MD, Eva Kanso, Yiying Tong
  - much more details and pointers to literature:
    - homology, cohomology, Hodge decomposition
    - living document... please help us improving it
- Another chapter in the course notes:
  - “Build Your Own DEC at Home” by Sharif Elcott & Peter Schröder