Discrete Exterior Calculus

How to Turn Your Mesh into a Computational Structure

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Applied Geometry Lab

Big Picture

Deriving a whole *Discrete Calculus*

- you need first a **discrete domain**
  - will induce the notion of chains
    - discrete representation of geometry
- then **discrete "differential" operators**
  - applied to our discrete geometric set-up
  - defined through cochains (discrete forms)

SCARED?

- **Don't be**: just numbers on mesh elmts!

Part I

Discrete
Geometric Setup

Discrete Setup

Starting with a **discrete domain**

- can be thought of as "approximation"
  - cell decomposition of smooth manifold

Nice simplicial mesh

- vertex, edge, triangles, ...

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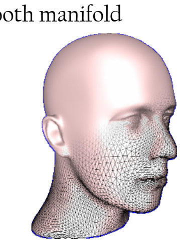
- vertex, edge, triangles, ...
- 2D domain
- curved 2D domain



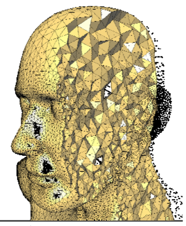
Discrete Setup

Starting with a **discrete domain**

- can be thought of as "approximation"
  - cell decomposition of smooth manifold

Nice simplicial mesh

- vertex, edge, triangles, ...
- 2D domain
- curved 2D domain
- 3D domain
- or even 4D and more!

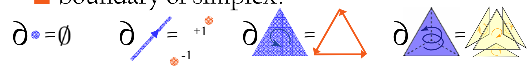


Quick Refresher Course

Notion of k-simplex

- non-degenerate convex hull of (k+1) vertices

- boundary of simplex?



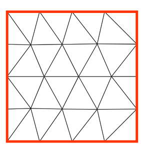
- simplicial complex
  - your typical triangle mesh/tet mesh

two k-simplices only intersect thru a common (k-1)-face

Discrete Subdomains

How to define geometric subsets?

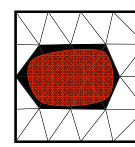
- again, the only "medium" we have is the mesh
  - simple example:

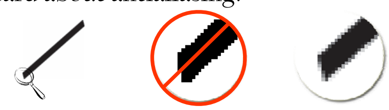


Discrete Subdomains

How to define geometric subsets?

- again, the only "medium" we have is the mesh
  - simple example:
    how to define this region?
  - "voxelized" is ok,
    but not great
- heard about antialiasing?



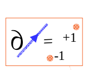


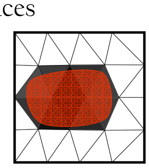
Notion of Chains

Allowing linear combination of simplices

- assign coefficients to simplices
  - not just 0 or 1
  - it's called a chain
    - we've seen it before;
      think boundary of an edge

$$c = \sum_{\sigma_i \in K} \alpha_i \sigma_i$$





Definition:

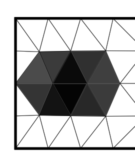
- k-chain = one value per k-simplex
  - think "column vector"
    - ... or simply an array of values in CS terms

Chains and Boundaries

Boundary of a chain is a chain:

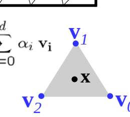
$$\partial c = \sum_{\sigma_i \in K^k} \alpha_i \, \partial\sigma_i = \sum_{\sigma_j \in K^{(k-1)}} \beta_j \, \sigma_j$$

- chains allow
  "sub-simplex" accuracy
  - anti-aliased version...



- chains extension of
  barycentric coordinates!

$$\mathbf{x} = \sum_{i=0}^{d} \alpha_i \, \mathbf{v}_i$$

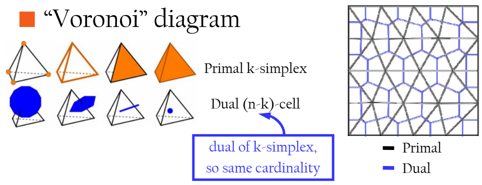


- notion of homology

Notion of Dual Complex

Associate to each k-simplex a (n-k)-cell

- connectivity of mesh *induces* another mesh
- "Voronoi" diagram



Chains are defined as well on dual mesh!

Part II

Turning our Mesh into a Computational Structure

Intrinsic Calculus on Meshes

Ready to bootstrap a discrete calculus

- using only values on simplices 
  - "measurements" in the domain
- preview:

point-based scalar field $\xrightarrow[d]{\nabla}$ *edge-based* vector field $\xrightarrow[d]{\nabla\times}$ *face-based* vector field $\xrightarrow[d]{\nabla\cdot}$ *cell-based* scalar field

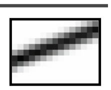
- deep roots in mathematics
  - exterior calculus, algebraic topology
- *but very simple to implement and use*

Discrete Differential Quantities

Mentioned repeatedly in the talks before...

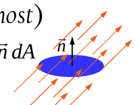
- they "live" at special places, as *distributions*
  - Gaussian curvature at vertices ONLY
  - mean curvature at edges ONLY
- they can be handled through integration
  - integration calls for k-***forms*** (antisymmetric tensors)
    - objects that **beg** to be integrated (ex: $\int f(x)dx$) that's a 1-form
  - k-forms are evaluated on kD set
    - 0-form is evaluated at a point, 1-form at a curve, etc...
    - what you can "measure"

Forms You Know For Sure

Digital Images: 2-forms 

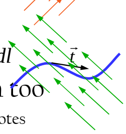
- incident flux on sensors (W/m^2)
- "hmm, looks like a chain, no?" (*almost*)

Magnetic Field ***B***: 2-form $\iint \vec{B}.\vec{n}\,dA$



- only measurement possible: flux!
- any physical flux is a 2-form too

Electrical Force ***E***: 1-form $\int \vec{E}.\vec{t}\,dl$



- any physical circulation is a 1-form too

notion of pseudo forms—see notes

Exterior Calculus of Forms

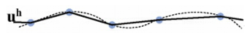
Foundation of calculus on smooth manifolds

- Historically, purpose was to extend div/curl/grad
  - Poincaré, Cartan, Lie, ...
- Basis of differential and integral computations
  - highlights topological and geometrical structures
  - modern diff. geometry, Hodge decomposition, ...
- A hierarchy of basic operators are defined:
  - d, $*$, $\wedge$, $\flat$, $\sharp$, i_X, $\mathcal{L}_X$
- See [Abraham, Marsden, Ratiu] , ch. 6-7

Discrete Forms?

Idea: *Sampling Forms on Each Simplex*

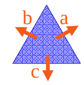
- extends the idea of point-sampling of fcts



- "sample" (i.e., integrate) a k-form on k-cells
- the rest is defined by linearity

$$\int_{\bigcup_j \sigma_j} \omega = \sum_j \int_{\sigma_j} \omega$$

- ex: if we know the flux on each edge, flux over the boundary of triangle is just the sum of the fluxes a+b+c



Discrete Forms As Cochains

Discrete k-form = values on each kD set

- primal forms on simplices, dual forms on cells
- **Not a chain**! Think "row vector" this time
  - in CS terms: k-form = array of values too

These discrete forms are cochains

- in math terms: chains pair with cochains (natural pairing = integration)
- if chain c valued $c[i]$ on σ_i & form ω valued $\omega[i]$ on σ_i ,

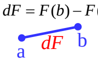
$$\langle \omega, c \rangle = \int_c \omega = \int_{\sum_i c[i]\sigma_i} \omega = \sum_i c[i] \int_{\sigma_i} \omega = \sum_j c[i]\omega[i]$$

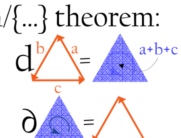
Notion of Exterior Derivative

Stokes/Green/{...} theorem: $\int_\sigma d\omega = \int_{\partial\sigma} \omega$

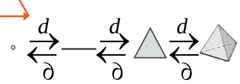
turns an integral into a boundary integral

$\int_a^b dF = F(b) - F(a)$





- d and ∂ are dual

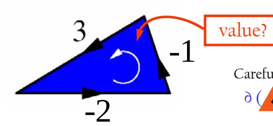


- Implementation?
  - As simple as an incidence matrix! $d = \partial^t$
  - ex: d(1-form)=incidence matrix of edges & faces

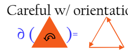
    Bean counting: array[|F|x|E|] x array|E| = array|F|

Exterior Derivative

Let's try



Careful w/ orientation!



- No "metric" needed! (no size measurement)
- Try d, then d again on an arbitrary form...
  - zero; why?
  - because $\partial \circ \partial = 0$
  - good: div(curl)=curl(grad)=0 *automatically*

Hodge Star

Take forms to dual complex *(and vice-versa)*

- switch values btw primal/dual

$$\star : \Omega^p \rightarrow \Omega_\star^{n-p}$$

- "diagonal" hodge star

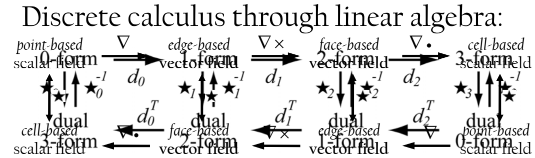
$$\frac{1}{|\star\sigma^p|}\int_{\star\sigma^p} \star\alpha^p = \frac{1}{|\sigma^p|}\int_{\sigma^p} \alpha^p$$

common "average" value

  - again, a simple (diagonal) matrix
- now the **metric** enters...
- Hodge star defines accuracy

Discrete deRham Complex

Discrete calculus through linear algebra:

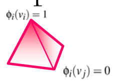


- simple exercise in matrix assembly
- all made out of two trivial operations:
  - summing values on simplices (d/∂)
  - scaling values based on local sizes (⋆)

Interpolating Discrete Forms

Whitney basis functions to interpolate forms

- 0-forms (functions)
  - linear "hat" functions
- 1-forms (edge elements!)
  - Whitney forms: $\phi_{ij} = \phi_i d\phi_j - \phi_j d\phi_i$
  - basis of 1-forms since $\phi_{ij}(e_{kl}) = 0$ on edges ≠ (i,j)
- 2-forms: face elmts
- 3-forms: constant per tet

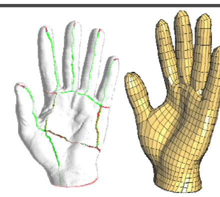


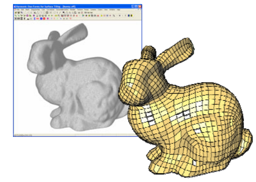
think ∇ for computations

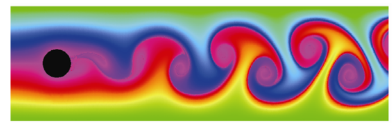
also a fct of ϕ and $\nabla\phi$

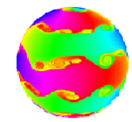
Higher order bases? See [Ke Wang et al. 2006]

What We'll Be Able To Do









Other Related Research Areas

Geometric Algebra

Mimetic Differencing

- same spirit, less geometry

Finite {Element|Volume|Differences}

Discrete Mechanics

- what about time discretization now?
- principle of *least-action* is crucial
  - motion is a geodesic if we use action as "metric"
- Last talk of the day

Take-Home Message

Geometric Approach to Computations

- discrete setup acknowledged from the get-go
- choice of proper habitat for quantities
- whole calculus built using only:
  - boundary of mesh elements
  - scaling by local measurements
- preserving *structural indentities*
  - they are not just abstract concepts: they represent defining symmetries

Join the Discrete World

Wanna know more about DEC?

- Chapter in course notes
  - "Discrete Differential Forms" by MD, Eva Kanso, Yiying Tong
  - much more details and pointers to literature:
    - homology, cohomology, Hodge decomposition
    - living document... please help us improving it
- Another chapter in the course notes:
  - "Build Your Own DEC at Home" by Sharif Elcott & Peter Schröder