What can we measure?

Eitan Grinspun, Columbia University
Overview

What characterizes shape?

- brief recall of classic notions
- how to express in discrete setting?

What structures are preserved?

- Gauss-Bonnet
- Minimal surfaces
- Steiner polynomial
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What structures are preserved?

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- Minimal surfaces
- Steiner polynomial
What to keep in mind

Where do quantities live?
  • consider going down parameter lane....

  • in the continuous setting, pointwise makes sense
What to keep in mind

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- consider going down parameter lane....
  \[ S(u, v) = (x(u, v), y(u, v), z(u, v)) \]

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What to keep in mind

Where do quantities live?

- consider going down parameter lane...
  \[ S(u, v) = (x(u, v), y(u, v), z(u, v)) \]

- in the continuous setting, pointwise sense

Attention

parameter-free zone
What to keep in mind
What to keep in mind

Where do quantities live?

“Pointwise notions considered harmful”
What to keep in mind

Where do quantities live?

“Pointwise notions considered harmful”

- quantities “live” on vertices, edges, or faces
- total quantity over a mesh neighborhood
What to keep in mind

Where do quantities live?

“\textit{Pointwise notions considered harmful}”

- quantities “live” on vertices, edges, or faces
- total quantity over a mesh \textit{neighborhood}
- wait…

\textit{isn’t living on a vertex a pointwise notion?}
What to keep in mind

Where do quantities live?

“Pointwise notions considered harmful”

- quantities “live” on vertices, edges, or faces
- total quantity over a mesh neighborhood
- wait…
  isn’t living on a vertex a pointwise notion?
- No. Total quantity over a mesh neighborhood.
Tangent Vector

Curve on surface, passing through point

recall:

Tangent, the first approximant

The limiting secant as the two points come together.
Discrete Tangent Vector

Curve on surface, passing through point

recall:

Tangent, the first approximant

The limiting secant as the two points come together.
Tangent Plane

All tangents at P lie on common plane

- Gives tangent *vector space*
Tangent Plane

All tangents at P lie on common plane
• Gives tangent vector space

vector addition
mult. by scalar
zero vector
e tc.
Metric

\[ g(v, w) = |v||w| \cos \angle(v, w) \]
in smooth, pointwise setting, the place to shop for “first-order” quantities

\[ g(v, w) = |v||w| \cos \angle (v, w) \]
Metric

\[ g(v, w) = |v||w| \cos \angle (v, w) \]

Length
- plug in \( v = w \)

Angle

Area

\[ g(v, v) = |v|^2 \]
**Metric**

\[ g(v, w) = 1 \cdot 1 \cdot \cos \angle (v, w) \]

**Length**
- plug in \( v = w \)

**Angle**
- use \( |v| = |w| = 1 \)

**Area**

\[ \cos^{-1} g(\hat{v}, \hat{w}) \]
Metric

\[ g(v, w) = |v||w| \cos \angle (v, w) \]

Length
- plug in \( v = w \)

Angle
- use \( |v| = |w| = 1 \)

Area
- parallelogram fixed by length and angle
Where do these live on a triangle mesh?

- Length
- Angle
- Area
Where do these live on a triangle mesh?

Length
  - 1 edge

Angle

Area
Where do these live on a triangle mesh?

Length
- 1 edge

Angle
- 2 edges

Area
Where do these live on a triangle mesh?

**Length**
- 1 edge

**Angle**
- 2 edges

**Area**
- 3 edges (the triangle)
Normal Vector

Perpendicular to tangent plane

- must choose orientation
Normal Sections

Special family of curves through point P
- choose any plane containing normal
- find the curve of plane/surface intersection
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Special family of curves through point P
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Normal Sections

Special family of curves through point P
  • choose any plane containing normal
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Sectional Curvature

Curvature of normal section
• curvature of surface in tangent direction

Recall smooth def’n:

Radius of curvature, \( r = \frac{1}{\kappa} \)

Recall discrete def’n:

Total signed curvature
\[ tsc(p) = \sum_{i=1}^{n} \alpha_i \]

Sum of turning angles.
Sectional Curvature

"Normal curvature"

Curvature of normal section
- curvature of surface in tangent direction

Recall smooth def’n:

Radius of curvature, \( r = \frac{1}{\kappa} \)

\[ \kappa = \frac{1}{r} \]

Curvature

Recall discrete def’n:

Total signed curvature

\[ tsc(p) = \sum_{i=1}^{n} \alpha_i \]

Sum of turning angles.
Principal Curvatures

Experiment:
rotate plane about normal,
plot sectional curvature

principal curvatures $K_1, K_2$
Principal Curvatures

Experiment:
rotate plane about normal,
plot sectional curvature

Caution
smooth case only

principal curvatures $k_1, k_2$
Principal Curvatures

Experiment:
rotate plane about normal,
plot sectional curvatures

$\kappa_1$ and $\kappa_2$ fully specify all sectional curvatures at $P$

Caution smooth case only

principal curvatures $\kappa_1, \kappa_2$
Mean & Gaussian Curvature

Elementary symmetric functions of $\kappa_1$, $\kappa_2$

- Gaussian curvature $K = \kappa_1 \kappa_2$
- mean curvature $H = \kappa_1 + \kappa_2$
Mean & Gaussian Curvature

Elementary symmetric functions of $\kappa_1, \kappa_2$

- Gaussian curvature $K = \kappa_1 \kappa_2$
- mean curvature $H = \kappa_1 - \kappa_2$

Gaussian and mean curvatures $(H$ and $K)$ fully specify all sectional curvatures at $P$
Mean & Gaussian Curvature

Elementary symmetric functions of $\kappa_1, \kappa_2$

- Gaussian curvature $K = \kappa_1 \kappa_2$
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Mean & Gaussian Curvature

Elementary symmetric functions of $\kappa_1$, $\kappa_2$

- Gaussian curvature $K = \kappa_1 \kappa_2$
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How to apply these pointwise definitions on a triangle mesh?

- we don’t have a smooth surface
- trouble at every corner (try evaluating $H^2$)
Mean & Gaussian Curvature

Elementary symmetric functions of $\kappa_1$, $\kappa_2$

- Gaussian curvature $K = \kappa_1 \kappa_2$
- Mean curvature $H = \kappa_1 + \kappa_2$

How to apply these pointwise definitions on a triangle mesh?

- We don’t have a smooth surface
- Trouble at every corner (try evaluating $H^2$)

Solution: look for key properties of $K$ and $H$
Gaussian Curvature

recall:

Gauß map, $\hat{n}(x)$

Point on curve maps to point on unit circle.
Gaussian Curvature

\[ K_p = \lim_{A \to 0} \frac{AG}{A} \]

recall:

Gauß map, \( \hat{n}(x) \)

Point on curve maps to point on unit circle
Preserve Gauss-Bonnet Theorem

Notion of integrated Gauss curvature as area of region on unit sphere

• Gauss-Bonnet Theorem
Preserve Gauss-Bonnet Theorem

**recall:**

**Turning number theorem**

\[ \int_{\Omega} \kappa \, ds = 2\pi k \]

For a closed curve, the integral of curvature is an integer multiple of \(2\pi\).
Preserve Gauss-Bonnet Theorem

Notion of integrated Gauss curvature as area of region on unit sphere

- Gauss-Bonnet Theorem

\[ 2\pi \chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA \]

\( 2-2g \) for closed, oriented surface

\( |f| - |e| + |v| \) for a simplicial complex
Gaussian Curvature

\[ K_p = \lim_{A \to 0} \frac{AG}{A} \]
Gaussian Curvature

\[ K_p = \lim_{A \to 0} \frac{A_G}{A} \]

On a mesh

- can’t take limit... but integral still makes sense
Gaussian Curvature

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On a mesh

- can’t take limit… but integral still makes sense
- apply Gauss map to vertex neighborhood
  - each face normal maps to a point
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- apply Gauss map to vertex neighborhood
  - each face normal maps to a point
  - each edge maps to an arc
  - vertex neighborhood maps to spherical polygon

\[ K_p = \lim_{{A \to 0}} \frac{A_G}{A} \]
Gaussian Curvature

\[ K_p = \lim_{A \to 0} \frac{A_G}{A} \]

On a mesh

- can’t take limit… but integral still makes sense
- apply Gauss map to vertex neighborhood
  - each face normal maps to a point
  - each edge maps to an arc
  - vertex neighborhood maps to spherical polygon
- our task:
  *compute area of spherical polygon*
Gaussian Curvature

Area of spherical polygon

\[ A = (2 - n) \pi + \sum_{i=1}^{n} \beta_i \]
Gaussian Curvature

Area of spherical polygon

\[ A = (2 - n)\pi + \sum_{i=1}^{n} \beta_i \]
Gaussian Curvature

Area of spherical polygon

\[ A = (2 - n) \pi + \sum_{i=1}^{n} \beta_i \]

*total* Gauss curvature at vertex
Gaussian Curvature

Area of spherical polygon

\[ A = (2 - n)\pi + \sum_{i}^{n} \beta_i \]

*total Gauss curvature at vertex*

- where do I find \( \beta_i \) on my mesh?
\[ \frac{\pi}{2} + (\pi - \alpha_i) + \frac{\pi}{2} + (\pi - \beta_i) = 2\pi \]

\[ \alpha_i = \pi - \beta_i; \quad \beta_i = \pi - \alpha_i \]
Gaussian Curvature

Area of spherical polygon

\[ A = (2 - n)\pi + \sum_{i=1}^{n} \beta_i \]
Gaussian Curvature

Area of spherical poly:

\[ A = (2 - n)\pi + \sum_{i=1}^{n} \beta_i \]
Gaussian Curvature

Area of spherical poly:

\[ A = (2 - n)\pi + \sum_{i}^{n} \beta_i \]

\[ A = (2 - n)\pi + n\pi - \sum_{i}^{n} \alpha_i \]
Gaussian Curvature

Area of spherical polyhedron

\[ A = (2 - n)\pi + \sum_{i}^{n} \beta_i \]

\[ A = \cancel{(2 - n)\pi} + n\pi - \sum_{i}^{n} \alpha_i \]

\[ A = 2\pi - \sum_{i}^{n} \alpha_i \]
Gaussian Curvature

Area of spherical poly

\[ A = (2 - n)\pi + \sum_{i=1}^{n} \beta_i \]

\[ A = (2 - n)\pi + n\pi - \sum_{i=1}^{n} \alpha_i \]

\[ A = 2\pi - \sum_{i=1}^{n} \alpha_i \]

*total Gauss curvature at vertex*
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied exactly
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

- Gauss-Bonnet
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied \textit{exactly}

- Gauss-Bonnet

\[ 2\pi \chi = \int_{\mathcal{S}} \kappa_{1} \kappa_{2} dA = \int_{\mathcal{S}} K dA \]
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied \textit{exactly}

- Gauss-Bonnet

\[ 2\pi \chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA \]

- discrete
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

• Gauss-Bonnet

\[ 2\pi \chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA \]

|f| - |e| + |v|

• discrete

\[ K_i = 2\pi - \sum_{t_{i,j,k}} \alpha_{jk} \]
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied \textit{exactly}

- Gauss-Bonnet
  \[ 2\pi \chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA \]

- discrete
  \[ K_i = 2\pi - \sum_{t_{ijk}} \alpha_{jk} \]

\[ \sum_i K_i = 2\pi(V - F/2) = 2\pi(F - 3F/2 + V) = 2\pi \chi \]
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

- Gauss-Bonnet

\[ 2\pi \chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA \]

\[ K_i = 2\pi - \sum_{t_{i,j,k}} \alpha_{j,k} \]

\[ \sum_i K_i = 2\pi (V - F/2) = 2\pi (F - 3F/2 + V) = 2\pi \chi \]

face angles sum to \( \pi \)
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied exactly

- Gauss-Bonnet

In discrete setting, it's easy to prove Gauss-Bonnet

\[ \sum_i K_i = 2\pi(V - F/2) = 2\pi(F - 3F/2 + V) = 2\pi \chi \]

face angles sum to \( \pi \)
Gaussian Curvature

Intrinsic curvature
• sees only in-plane angles
• does not depend on embedding

Discrete setting
• only pedestrian calculations required to evaluated, and to prove Gauss-Bonnet
• associated to vertex neighborhood

think total Gauss curvature near vertex

\[ A = 2\pi - \sum_{i}^{n} \alpha_i \]
Mean Curvature \((k_1 + k_2)\)

Variational structure of mean curvature

- surfaces which minimize area
  - soap bubbles
- at any given point:
  - \(k_1 = -k_2\)
  - \(H = 0\)
  - \(H = H N = 0\)
Mean Curvature \((\kappa_1 + \kappa_2)\)

Variational structure of mean curvature

- surfaces which minimize area
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Variational structure of mean curvature

- surfaces which minimize area
  - soap bubbles
- at any given point:
  - \(\mathbf{H} = 0\)
  - \(\mathbf{H} = \mathbf{HN} = 0\)

area is (locally) minimum iff mean curvature is zero

minimize \(A\)

\[ \text{solve } \mathbf{H} = 0 \]
Mean Curvature Vector

\[ \mathbf{H} = \text{grad area} \]

Calculus of Variations

- stationary area \( \leftrightarrow \) grad area \( = \mathbf{H} = 0 \)
Mean Curvature Vector

\[ \vec{H} = \text{grad area} \]

Calculus of Variations

- stationary area \( \Leftrightarrow \text{grad area} = \vec{H} = 0 \)

some prefer \( H = -\text{grad area} \)
Mean Curvature Vector

$\vec{H} = \text{grad area}$

$\text{area} = bh$
Mean Curvature Vector

\[ \vec{H} = \text{grad area} \]

\[ 2 \times \text{area} = bh \]
Mean Curvature Vector

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\[ (\text{grad area}) \; v = \Delta \text{area} \]
Mean Curvature Vector

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\[ 2 \times \text{area} = bh \]

\[ \Delta \text{area} = \]

\[ (\text{grad area}) v = \Delta \text{area} \]
Mean Curvature Vector

\[ \vec{H} = \text{grad area} \]

\[
\begin{align*}
2 \times \text{area} &= bh \\
\Delta \text{area} &= b \Delta h
\end{align*}
\]

\[(\text{grad area}) \cdot \mathbf{v} = \Delta \text{area}\]
Mean Curvature Vector

\[ \vec{H} = \text{grad area} \]

\[ 2 \times \text{area} = bh \]
\[ \Delta \text{area} = b \Delta h \]
\[ = b (\vec{t} \cdot \vec{n}) \]

(\text{grad area}) \vec{n} = \Delta \text{area}
Mean Curvature Vector

\[ \hat{H} = \text{grad area} \]

\[
2 \times \text{area} = bh \\
\Delta \text{area} = b \Delta h \\
= b (\hat{\tau} \cdot \nabla) \\
\text{let } \hat{\tau} = b \hat{t} \\
\]

(\text{grad area}) \cdot v = \Delta \text{area}
Mean Curvature Vector

\[ \vec{H} = \text{grad area} \]

\[ 2 \times \text{area} = bh \]

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\[ = b (\hat{\varepsilon} \cdot \vec{v}) \]

\[ \text{let } \hat{\varepsilon} = b \hat{t} \]

\[ = \vec{b} \cdot \vec{v} \]

(\text{grad area}) \ \vec{v} = \Delta \text{area}
Mean Curvature Vector

\[ \overrightarrow{H} = \text{grad area} \]

2x area = bh
\[ \Delta \text{area} = b \Delta h = b (\hat{\varepsilon} \cdot \nabla) \]
let \[ \hat{\varepsilon} = b \hat{\varepsilon} \]
\[ = \hat{b} \cdot \nabla \]

(\text{grad area}) \ \vec{v} = \Delta \text{area}
Mean Curvature Vector

Evaluation

• sum contributions around each vertex
Mean Curvature Vector

Evaluation

- sum contributions around each vertex
Mean Curvature Vector

Evaluation

- sum contributions around each vertex

\[ 2H_i = \sum_j H_{e_{ij}} = 2\nabla_i A \]
\[ = \sum_j (\cot \alpha_{ij} + \cot \alpha_{ji})(p_i - p_j) \]

“cotan formula”
[Pinkall & Polthier]
Curvature Measures a la Steiner

Steiner, Cauchy, Hadwiger
- expand a convex set outward by epsilon

\[ \mathcal{K}_\varepsilon = \{ x \in \mathbb{R}^n : d(\mathcal{K}, x) \leq \varepsilon \} \]
A Steiner walk-through, 2d

$A' = A + \ldots$

Inflate a planar polygon by epsilon

What is the new area?
A Steiner walk-through, 2d

\[ A' = A + \sum_{i} \epsilon a_i + \ldots \]

Each edge contributes a rectangle
A Steiner walk-through, 2d

\[ A' = A + \sum_i \varepsilon a_i + \sum_j \varepsilon^2 \theta_j \]

Each vertex contributes a sector
A Steiner walk-through, 3d

\[ V' = V + \ldots \]

Inflate a polyhedron

What is the new volume?
A Steiner walk-through, 3d

\[ V' = V + \varepsilon \sum_{i} A_i \]

Each face contributes a parallelootope
A Steiner walk-through, 3d

\[ V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j \]

Each edge contributes a wedge of a cylinder
A Steiner walk-through, 3d

\[ V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \epsilon^3 \sum_l K_l \]

Each vertex contributes a spherical wedge
A Steiner walk-through, 3d

\[ V' = V \]

\[ K = 2\pi - \sum_m \alpha_m \]

\[ + \epsilon^2 \sum_j a_j \theta_j \]

\[ + \epsilon^3 \sum_l K_l \]

Each vertex contributes a spherical wedge
A Steiner walk-through, 3d

\[ V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \epsilon^3 \sum_l K_l + \epsilon^3 2\pi \chi \]

by Gauss-Bonnet

Each vertex contributes to the total wedge
Inflation in Smooth Setting

Inflate smooth surface, measure swept area

\[ \varepsilon c_1 \int_S H_0 \, dA \]
\[ + \varepsilon^2 c_2 \int_S H_1 \, dA \]
\[ + \varepsilon^3 c_3 \int_S H_2 \, dA \]

\[ H_0 = 1, \quad H_1 = (\kappa_1 + \kappa_2), \quad H_2 = \kappa_1 \kappa_2 \]
**Inflation in Smooth Setting**

In inflate smooth surface, measure swept area

\[ \epsilon c_1 \int_S H_0 dA \]

\[ + \epsilon^2 c_2 \int_S H_1 dA \]

\[ + \epsilon^3 c_3 \int_S H_2 dA \]

**Total Area**

\[ H_0 = 1, \quad H_1 = (\kappa_1 + \kappa_2), \quad H_2 = \kappa_1 \kappa_2 \]

---

**A Steiner walk-through, 3d**

\[ V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \epsilon^3 \sum_l K_l \]

Each vertex contributes a spherical wedge
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A Steiner walk-through, 3d

\[ V' = V + \varepsilon \sum_i A_i + \varepsilon^2 \sum_j a_j \theta_j + \varepsilon^3 \sum_l K_l \]

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\( H_0 = 1, \quad H_1 = (\kappa_1 + \kappa_2), \quad H_2 = \kappa_1 \kappa_2 \)
Vertex contributes a spherical wedge

\[ V' = V + \varepsilon \sum_i A_i + \varepsilon^2 \sum_j a_j \theta_j + \varepsilon^3 \sum_l K_l \]

\[ \varepsilon c_1 \int_S H_0 dA \]
\[ \varepsilon^2 c_2 \int_S H_1 dA \]
\[ \varepsilon^3 c_3 \int_S H_2 dA \]

\[ H_0 = 1, \quad H_1 = (\kappa_1 + \kappa_2), \quad H_2 = \kappa_1 \kappa_2 \]
V' = V + \varepsilon \sum_i A_i + \varepsilon^2 \sum_j a_j \theta_j + \varepsilon^3 \sum_l K_l

\varepsilon c_1 \int_S H_0 dA
\varepsilon^2 c_2 \int_S H_1 dA
\varepsilon^3 c_3 \int_S H_2 dA

H_0 = 1, \ H_1 = (\kappa_1 + \kappa_2), \ H_2 = \kappa_1 \kappa_2
$$V' = V + \varepsilon \sum_i A_i + \varepsilon^2 \sum_j a_j \theta_j + \varepsilon^3 \sum_l K_l$$

$H_0 = 1$, $H_1 = (\kappa_1 + \kappa_2)$, $H_2 = \kappa_1 \kappa_2$
$V' = V$

$+ \varepsilon \sum_i A_i$

$+ \varepsilon^2 \sum_j a_j \theta_j$

$+ \sum_l K_l$

$\varepsilon c_1 \int_S H_0 dA$

$\varepsilon^2 c_2 \int_S H_1 dA$

$\varepsilon c_3 \int_S H_2 dA$

Vertex contributes a spherical wedge

$H_0 = 1$, $H_1 = (\kappa_1 + \kappa_2)$, $H_2 = \kappa_1 \kappa_2$

Length $\times$ angle
\[ V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \sum_l K_l \]

Vertex contributes a spherical wedge:

\[ H_0 = 1, \quad H_1 = (\kappa_1 + \kappa_2), \quad H_2 = \kappa_1 \kappa_2 \]
Life & Times of Mean Curvatures

<table>
<thead>
<tr>
<th>Structure</th>
<th>variational (area)</th>
<th>Steiner polynomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>Species</td>
<td>vector</td>
<td>scalar</td>
</tr>
<tr>
<td>Habitat</td>
<td>vertices</td>
<td>edges</td>
</tr>
<tr>
<td>Expression</td>
<td>cotan formula</td>
<td>length $\times$ dihedral angle</td>
</tr>
</tbody>
</table>

![Diagram of a geometric structure](image-url)
A Plug for Intrinsic Measures

Axiomatic approach
• “what is a reasonable measure?”
• straightforward application to parallelotopes

Geometric probability
• geometry as a dart throwing game

Theorem Hadwiger (1957)
• “These are the only measures you should care about”

*Certain restrictions may apply.* Not responsible for exaggerated or untrue claims. If you are elderly, pregnant, or alive, please ask your doctor before using Hadwiger’s theorem. Not responsible for incidental, consequential, or any other damages. If you are reading this, you are not paying enough attention to the talk. Stop reading this and listen to me.
What is a reasonable measure?

Properties

• a measure is scalar-valued \( \mu(S) \in \mathbb{R} \)
• empty set \( \mu(\emptyset) = 0 \)
• additivity \( \mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B) \)
• normalization (parallelootope, \( P \))
  • example: volume

\[ \mu_n(P) = x_1x_2x_3 \ldots x_n \]

Other measures?
Invariant Measures

Intrinsic volumes
  • n measures in n dimensions
  • how to generalize to compact convex sets?

Geometric probability
  • measure points in set
  • probability of hitting set
Geometric probability

Blindly throw darts... count number of hits
Darts: k-dim subspaces of n-D

- points
- lines
- planes
- volumes

\( k=0 \)
\( n=2 \)
Geometric probability

Indicator function, $X_C(\omega_i)$

- input: a dart, $\omega$
- output (point dart):
  1 if dart hits body
  0 if dart misses body

$k=0$
$n=2$
Geometric probability

Indicator function, $X_C(\omega_i)$

- input: a dart, $\omega$
- output (point dart):
  1 if dart hits body
  0 if dart misses body
- in general, output is # hits
Geometric probability

Throw $N$ random darts to estimate area

\[
\frac{A_C}{A_R} \sim \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)
\]

$k=0$

$n=2$
Geometric probability

Throw $N$ random darts to estimate area
Throw all the darts you have...

$$\frac{A_C}{A_R} \sim \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$

$$A_C \propto \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$
Geometric probability

Throw $N$ random darts to estimate area
Throw all the darts you have...

$A_C \propto \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$

$2^{\text{nd}}$ intrinsic volume of $C$

$\mu_2(C)$

Measure of darts ($k=0,n=2$)

$\lambda_0^2(C)$

Hitting target $C$
Examples of dart-throwing

- Measure of lines through rectangle gives surface area.
- Measure of planes through polyline gives length.
- Measure of planes through polytope gives mean width.
Hadwiger (1957)

FUNDAMENTAL RESULT
Hadwiger (1957)

These measures form a basis for all continuous, additive, rigid motion invariant measures on ring of convex sets.

FUNDAMENTAL RESULT
Questions to take home

What can we measure?
- length, angle, area, Gauss & mean curvatures

Where does it live?
- vertex (one-ring), edge (flaps), face

What is its type?
- scalar, vector, tensor...

What structure does it preserve?
- Gauss-Bonnet, area variation, Steiner polynomial
Further Reading

**Smooth**
*Geometry and the Imagination*
by Hilbert and Cohn-Vossen

**Discrete**
*DDG Course Notes* chapters 1–3

“Introduction to DDG” [Grinspun and Secord]

“What can we measure?” [Schröder]

“Curvature measures for discrete surfaces” [Sullivan]
Overview

What characterizes shape?
- brief recall of classic notions
- how to express in discrete setting?

What structures are preserved?
- Gauss-Bonnet
- Minimal surfaces
- Steiner polynomial

Normal Sections

Special family of curves through point P
- choose any plane containing normal
- find the curve of plane/surface intersection

Gaussian Curvature

Area of spherical poly
\[ A = (2 - \mu) \pi + \sum_{i=1}^{n} \beta_i \]
\[ A = (2 - \mu) \pi + \pi r^2 - \sum_{j=1}^{n} \alpha_j \]
\[ A = 2\pi - \sum_{j=1}^{n} \alpha_j \]

Mean Curvature \( (\kappa_1 + \kappa_2) \)

Variational structure of mean curvature
- surfaces which minimize area
- soap bubbles
- at any given point:
  - \( \kappa_1 = \kappa_2 \)
  - \( \mu = 0 \)
  - \( H = HN = 0 \)

Mean Curvature Vector

Evaluation
- sum contributions around each vertex
\[ 2H_i = \sum_j H_{ij} = 2 \nabla_i A \]
\[ = \sum_j (\cot \alpha_{ij} + \cot \alpha_{ji})(p_i - p_j) \]

"cotan formula" [Pinkall & Polthier]

A Steiner walk-through, 2d

\[ A' = A + \sum_i \epsilon \alpha_i \]
\[ + \sum_j \epsilon^2 \theta_j \]

Each vertex contributes a sector

Life & Times of Mean Curvatures

Structure: variational (area) Steiner polynomial
Species: vector scalar
Habitat: vertices edges
Expression: cotan formula length \times dihedral angle

Examples of dart-throwing

Measure of planes through polyline gives length
Measure of planes through polytope gives mean width