What can we measure?

Eitan Grinspun, Columbia University

Overview

What characterizes shape?

- brief recall of classic notions
- how to express in discrete setting?

What structures are preserved?

- Gauss-Bonnet
- Minimal surfaces
- Steiner polynomial

Overview

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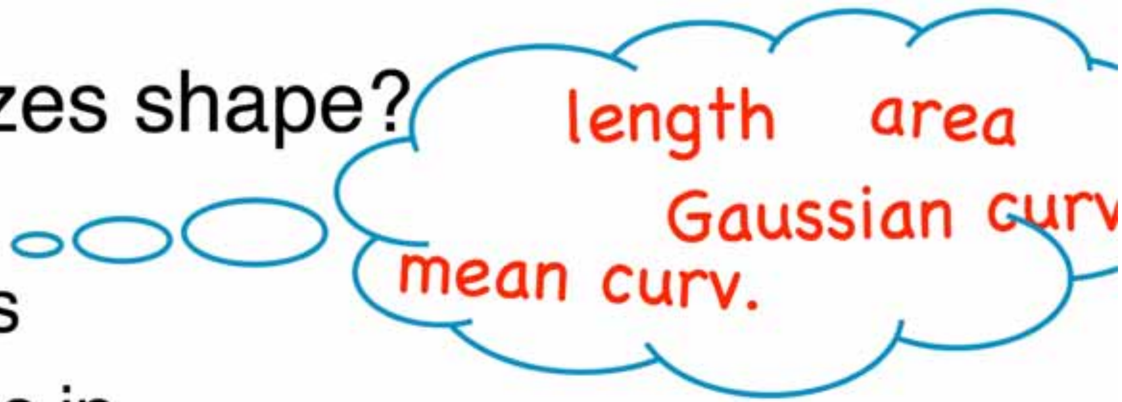

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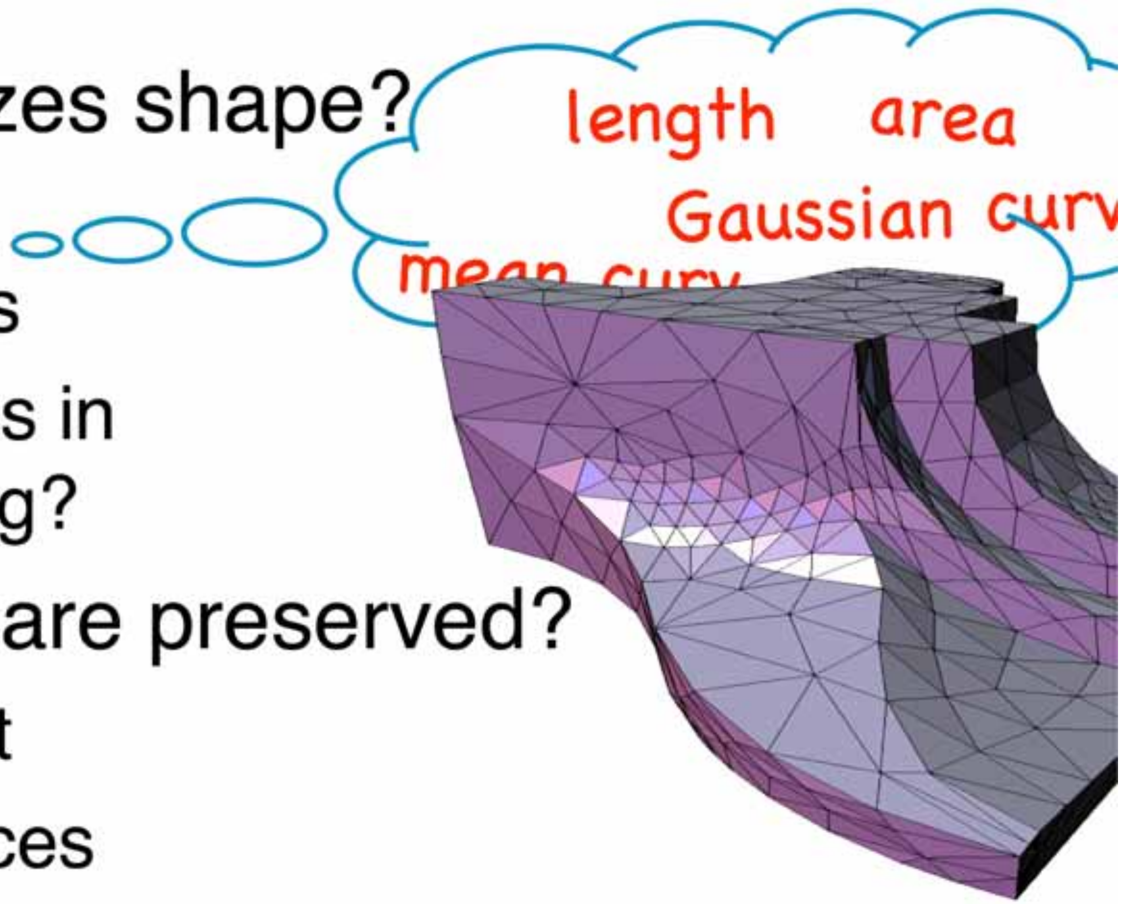

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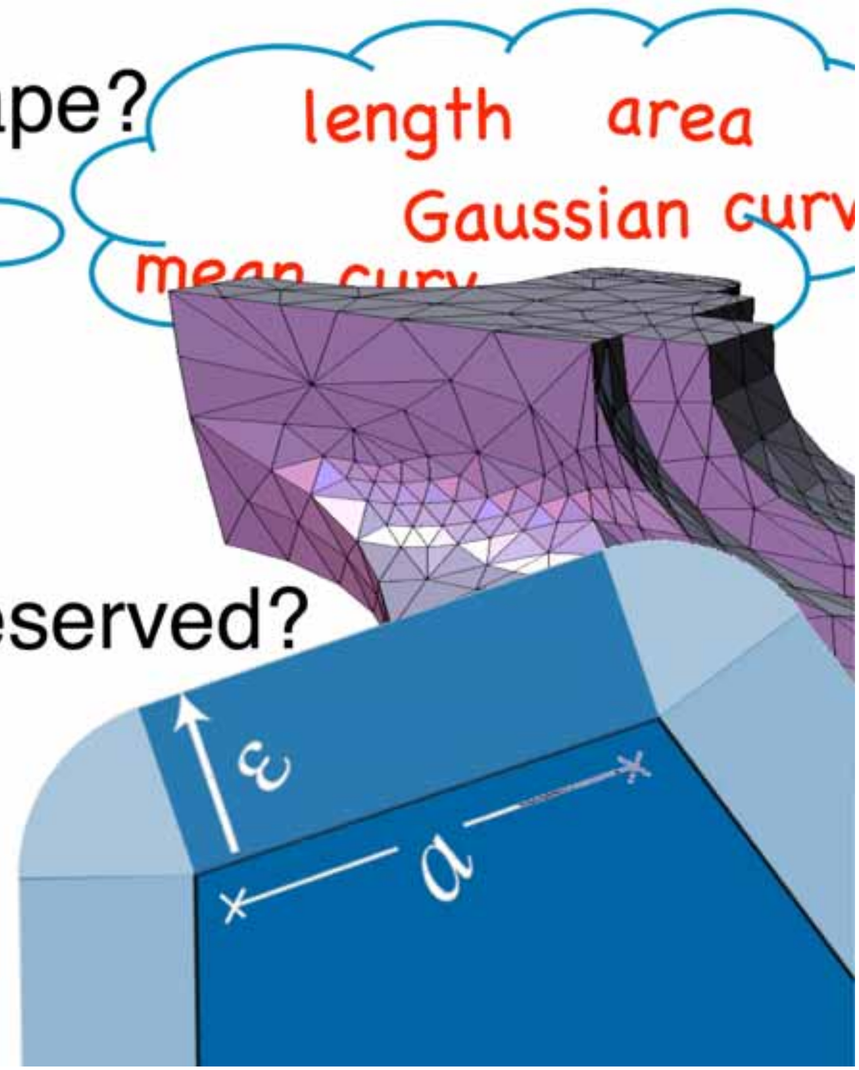

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What to keep in mind

Where do quantities live?

- consider going down parameter lane….

- in the continous setting, pointwise makes sense

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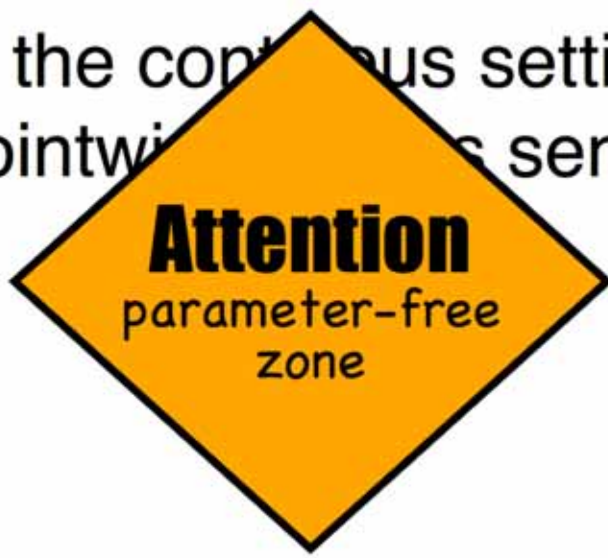

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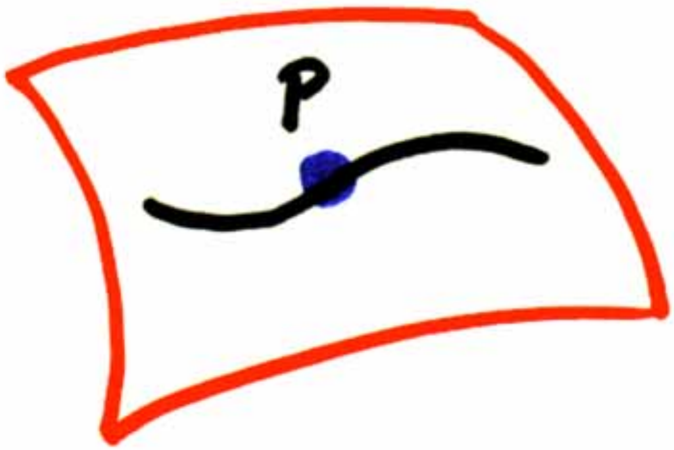
- No. Total quantity over a mesh
  neighborhood.

Tangent Vector

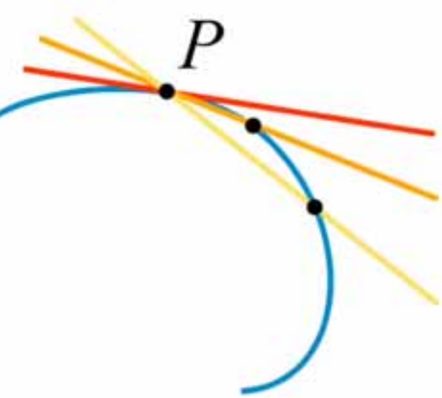
Curve on surface, passing through point



recall:

Tangent, the first approximant

The limiting secant as the two points come together.



Discrete

Tangent Vector

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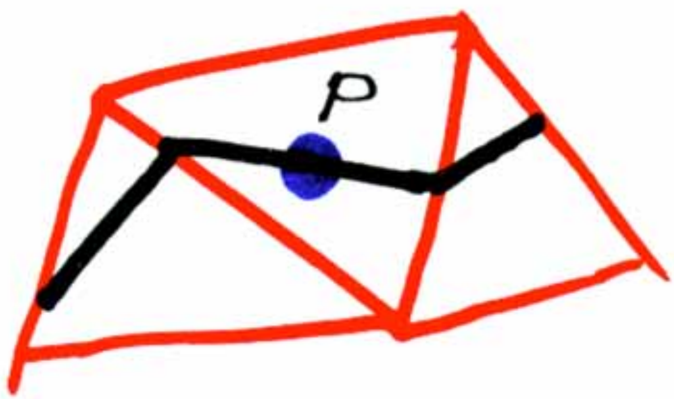

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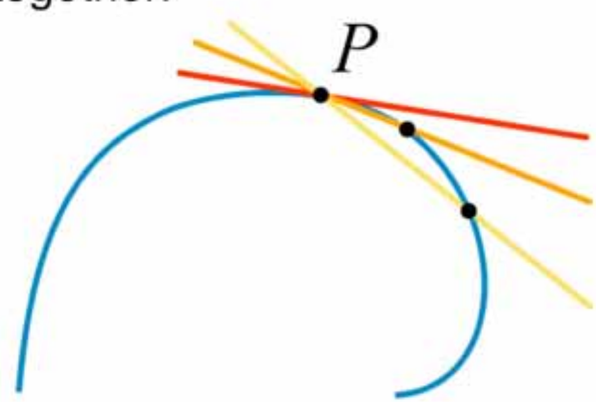
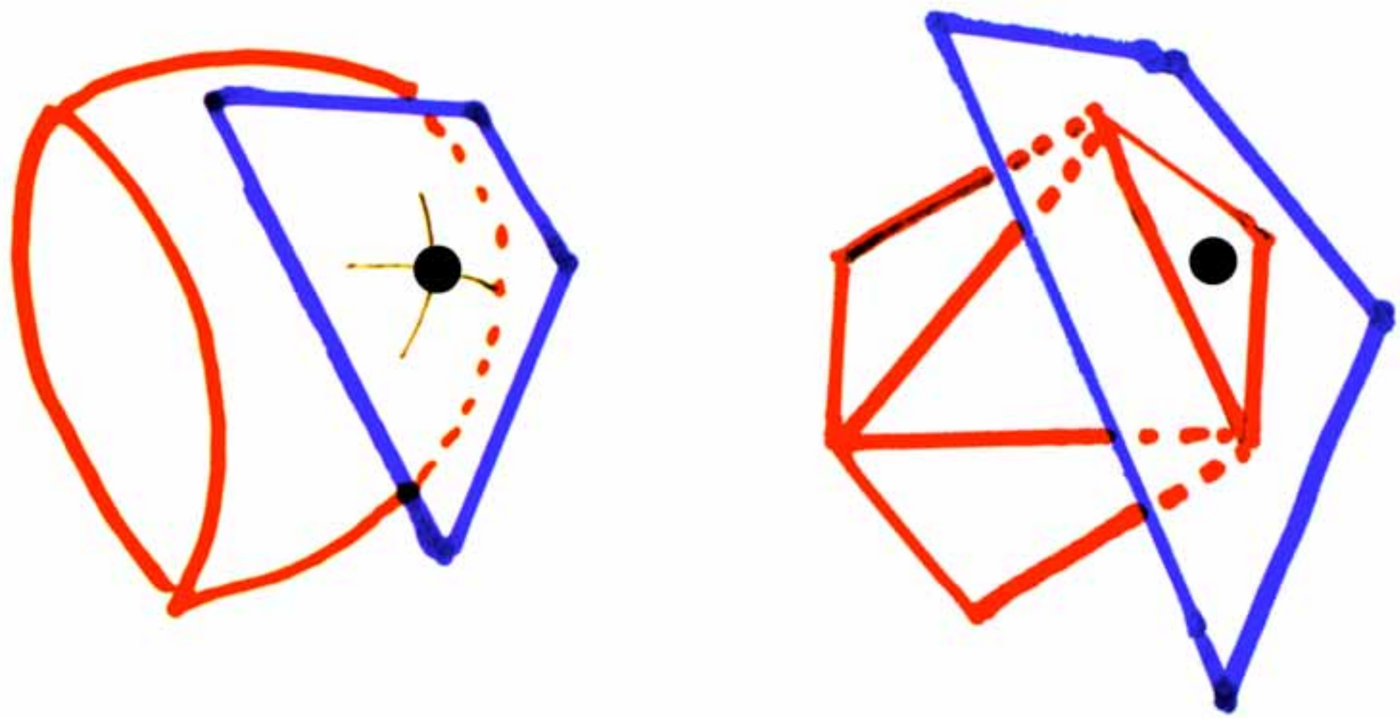


Tangent Plane

All tangents at P lie on common plane

- Gives tangent *vector space*



vector addition
mult. by scalar
zero vector
etc.

Tangent Plane

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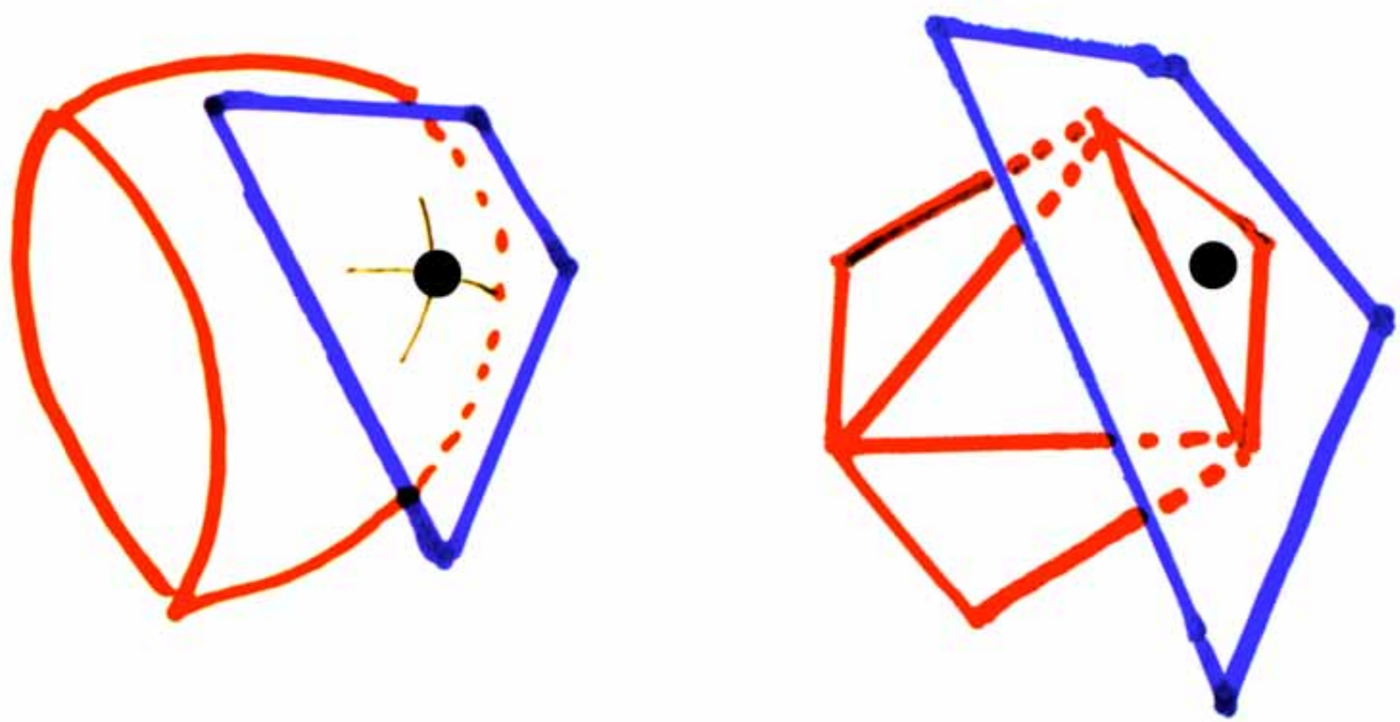


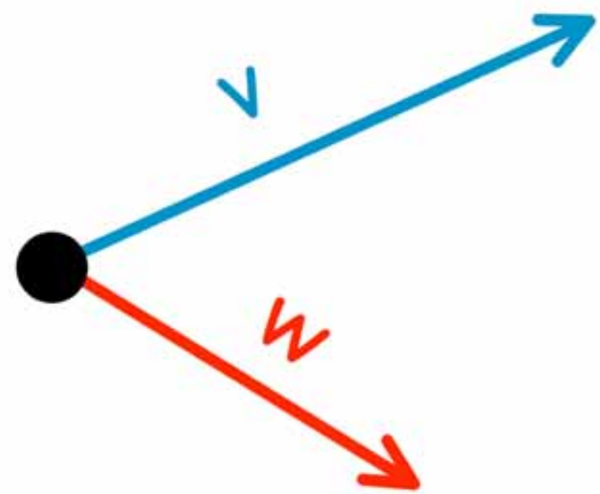
Metric

$$g(v, w) = |v||w| \cos \angle(v, w)$$

Length

Angle

Area



Metric

in smooth, pointwise setting, the place to shop for "first-order" quantities

$$g(v, w) = |v||w| \cos \angle(v, w)$$

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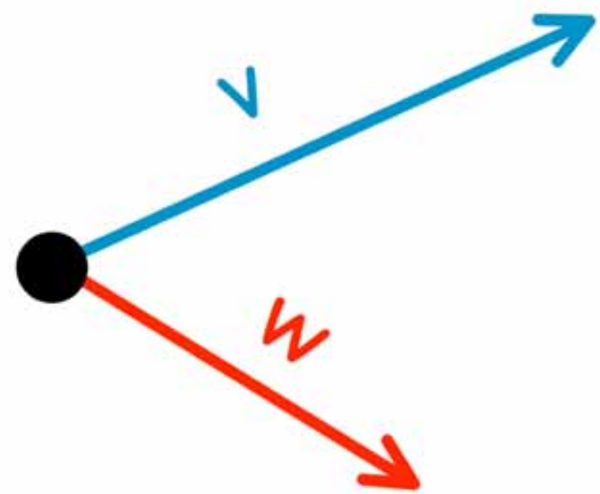

Metric

$$g(v,w) = |v||w| \cos \angle(v,w)$$

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- plug in v=w

Angle

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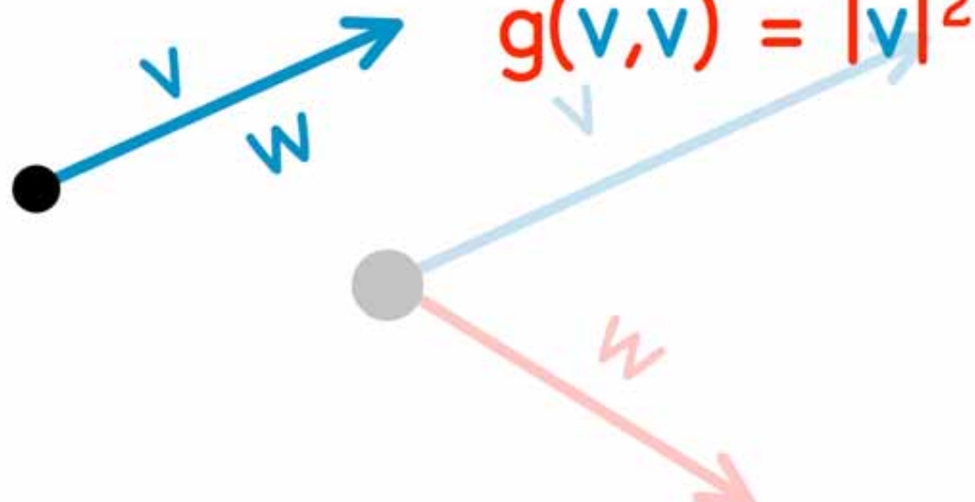


Metric

$$g(v, w) = |1| \cdot |1| \cdot \cos \angle (v, w)$$

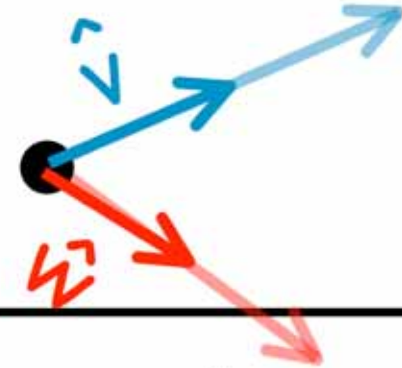
Length

- plug in v=w

Angle

- use |v|=|w|=1

Area



$$\cos^{-1} g(\widehat{v}, \widehat{w})$$

Metric

$$g(v, w) = |v||w| \cos \angle(v, w)$$

Length

- plug in v=w

Angle

- use |v|=|w|=1

Area

- parallelogram fixed by length and angle

Discrete

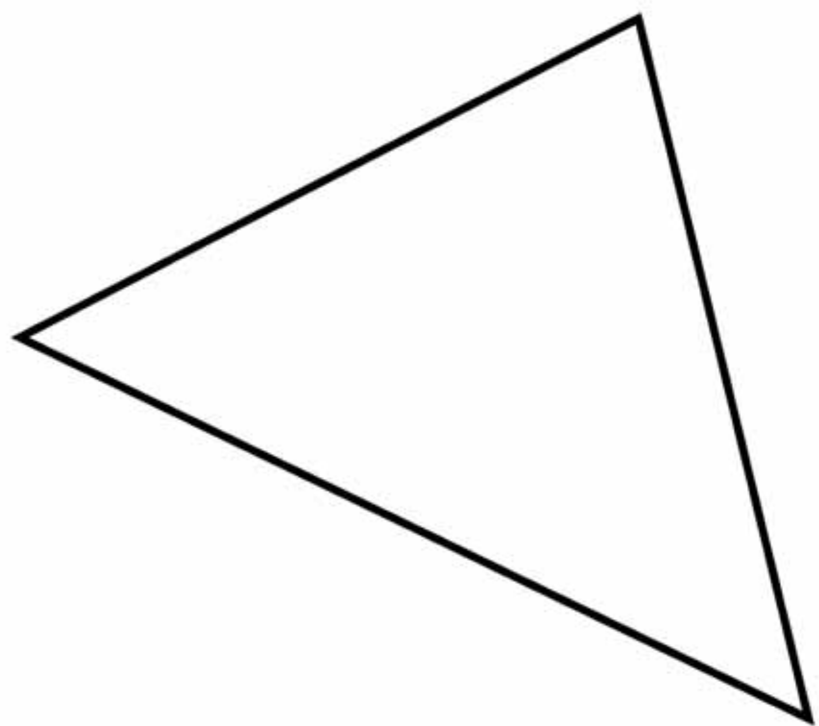
Metric

Where do these live on a triangle mesh?

Length

Angle

Area



Discrete

Metric

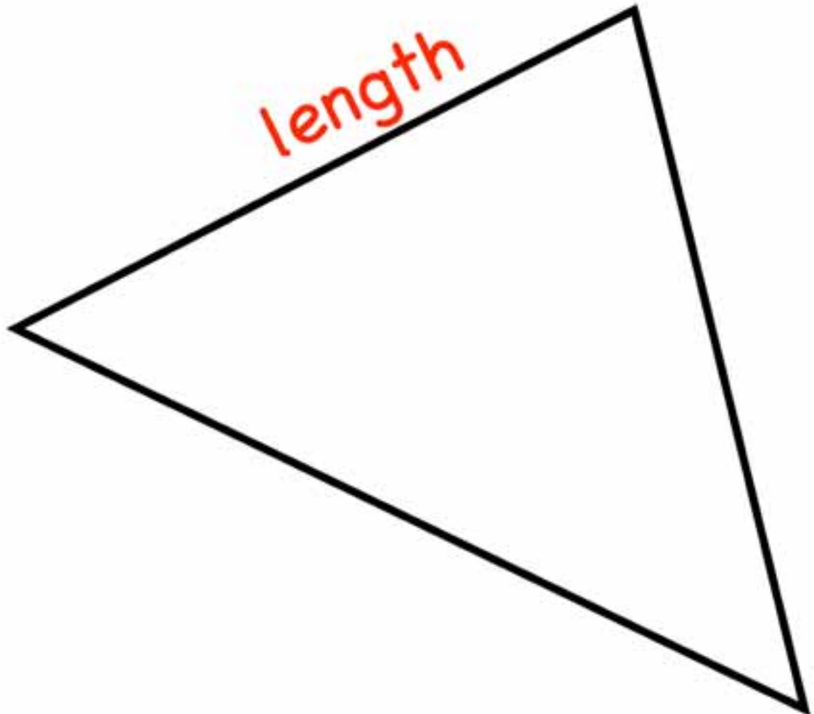
Where do these live on a triangle mesh?

Length

- 1 edge

Angle

Area



Discrete

Metric

Where do these live on a triangle mesh?

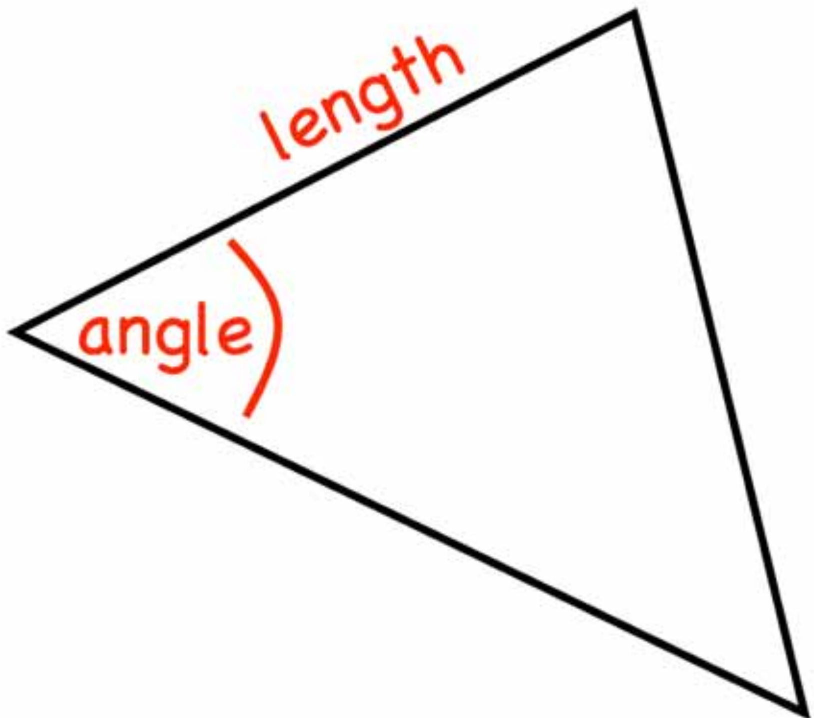
Length

- 1 edge

Angle

- 2 edges

Area



Discrete

Metric

Where do these live on a triangle mesh?

Length

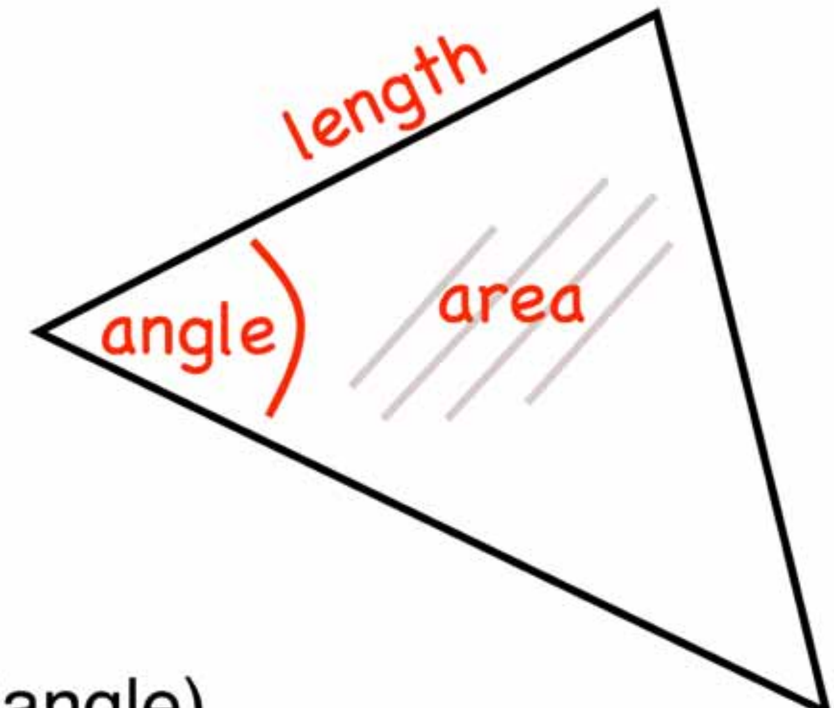
- 1 edge

Angle

- 2 edges

Area

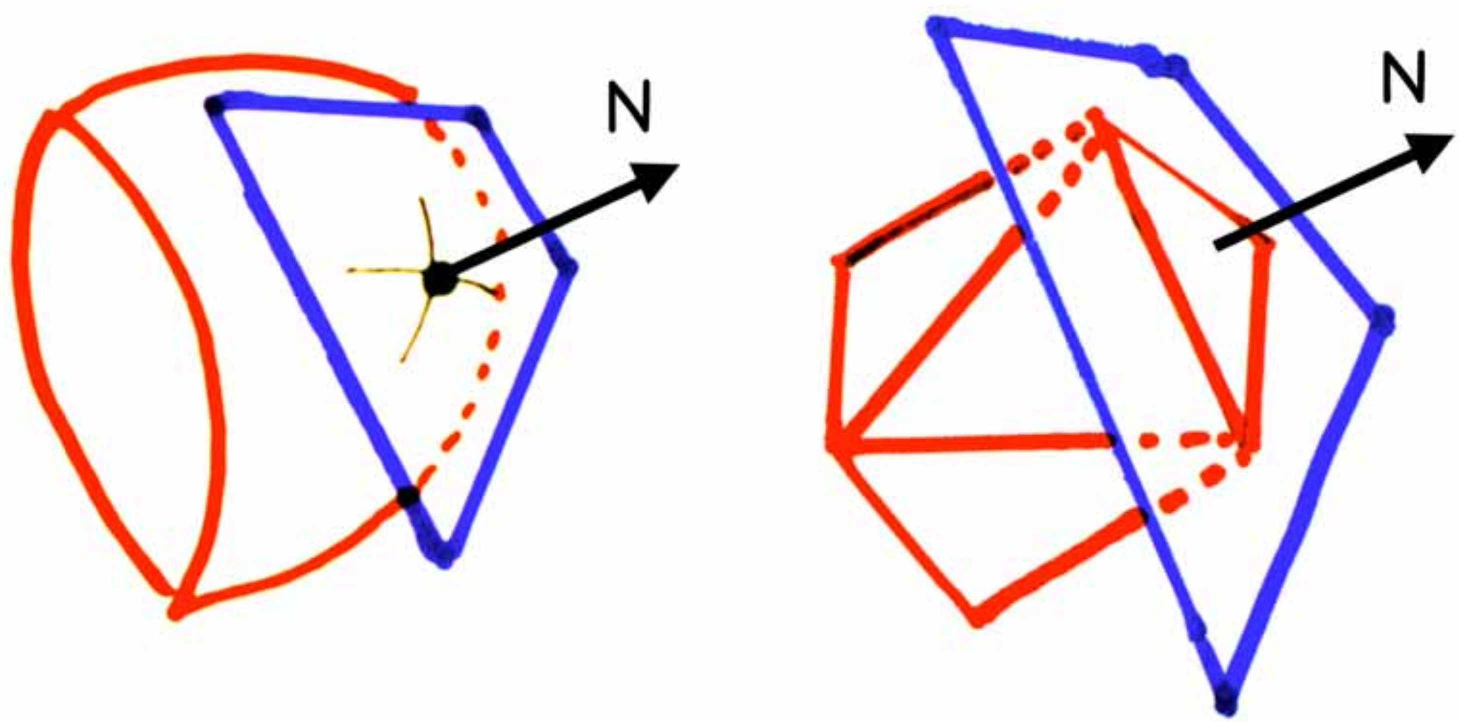
- 3 edges (the triangle)



Normal Vector

Perpendicular to tangent plane

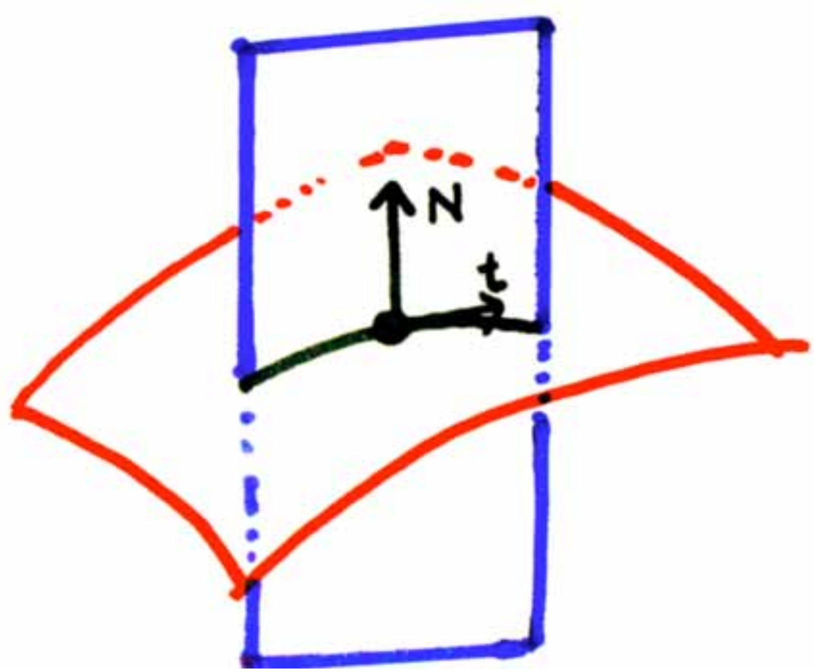
- must choose orientation



Normal Sections

Special family of curves through point P

- choose any plane containing normal
- find the curve of plane/surface intersection



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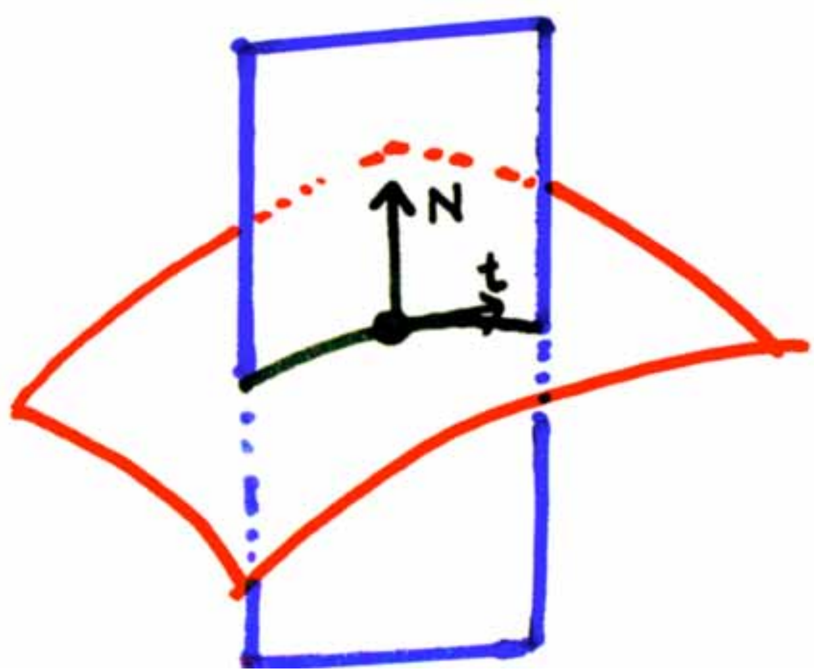

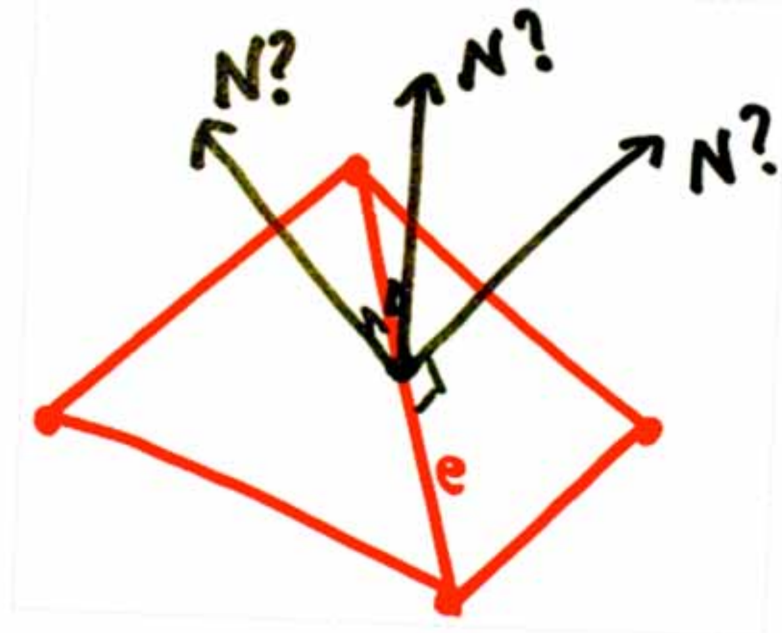

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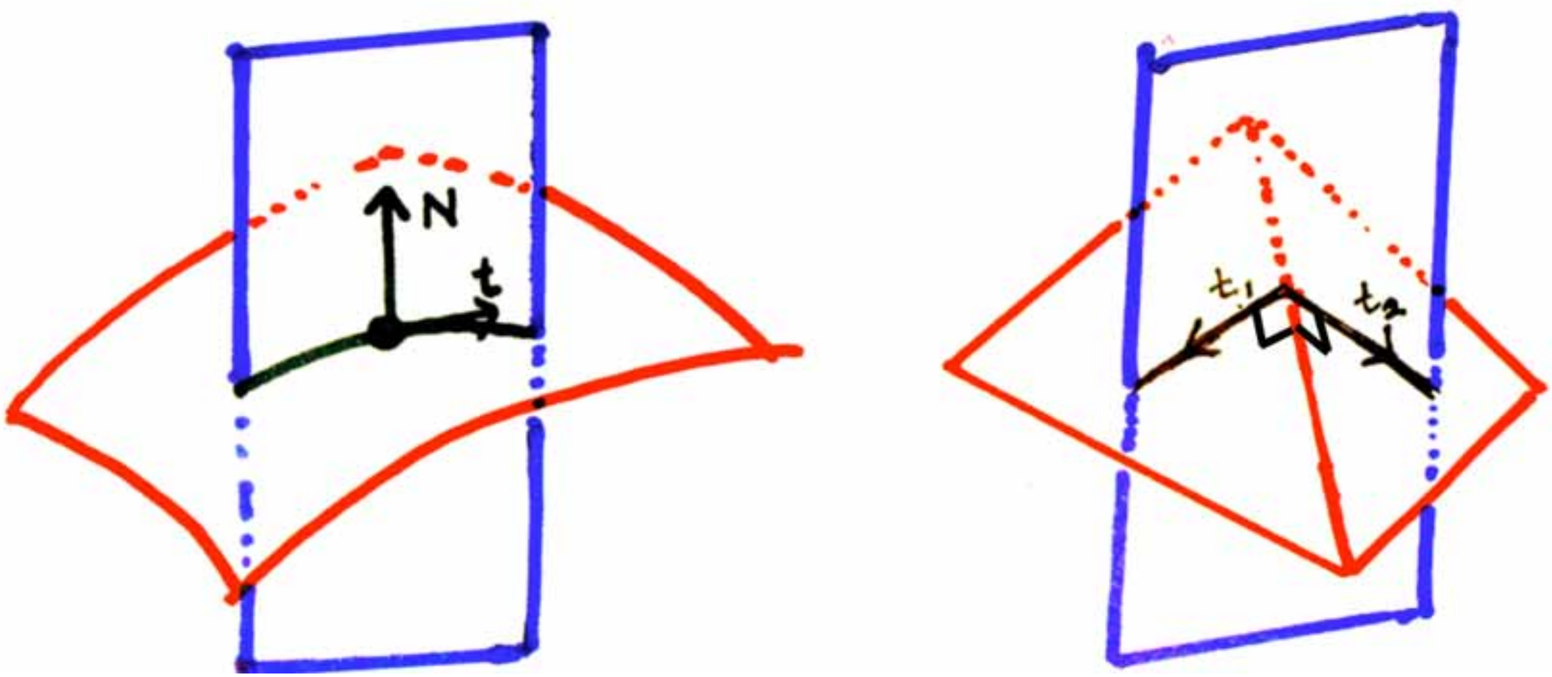


Sectional Curvature

Curvature of normal section

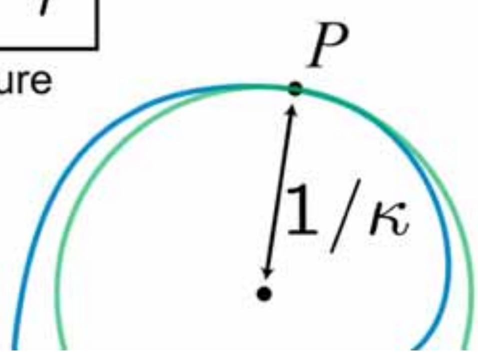
- curvature of surface *in tangent direction*

recall smooth def'n:

Radius of curvature, $r = 1/\kappa$

$$\kappa = \frac{1}{r}$$

Curvature

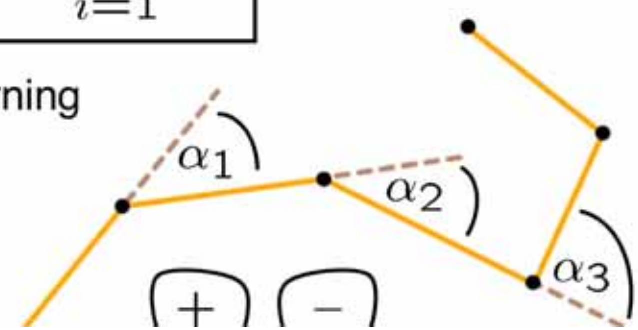


recall discrete def'n:

Total signed curvature

$$tsc(p) = \sum_{i=1}^{n} \alpha_i$$

Sum of turning angles.



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a.k.a. "Normal curvature"

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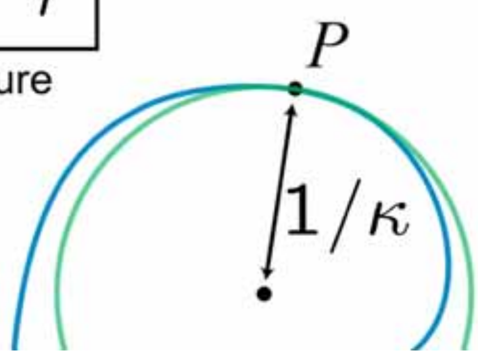

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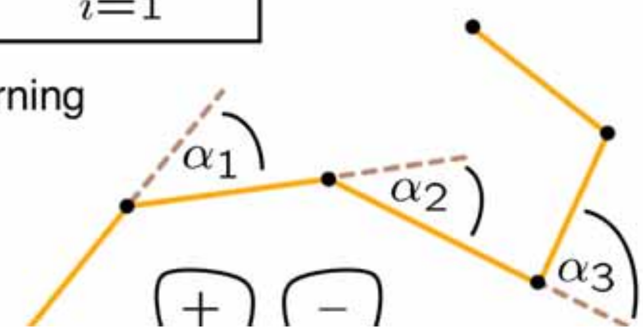
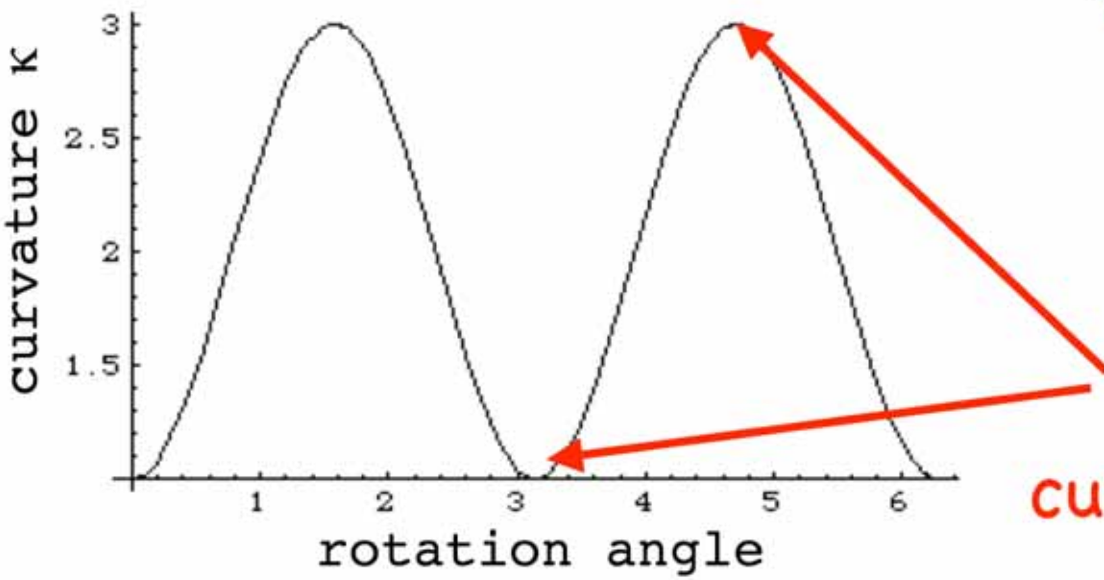


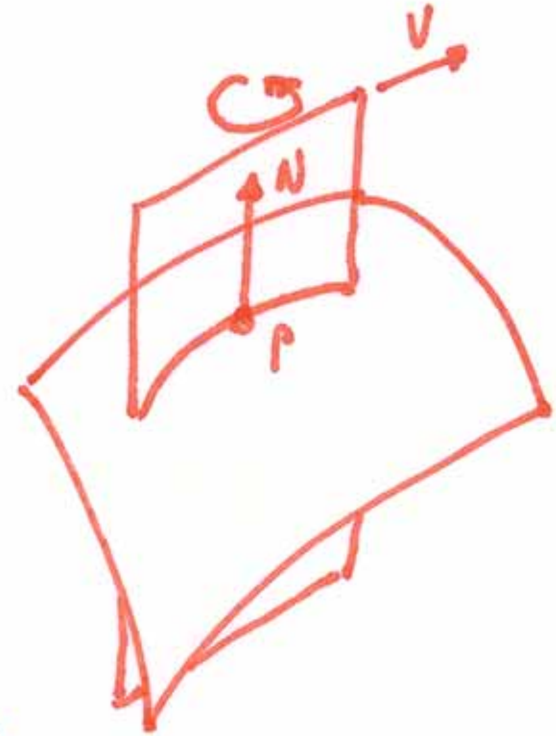
Principal Curvatures

Experiment:

rotate plane about normal,

plot sectional curvature





Principal Curvatures

Caution
smooth case only

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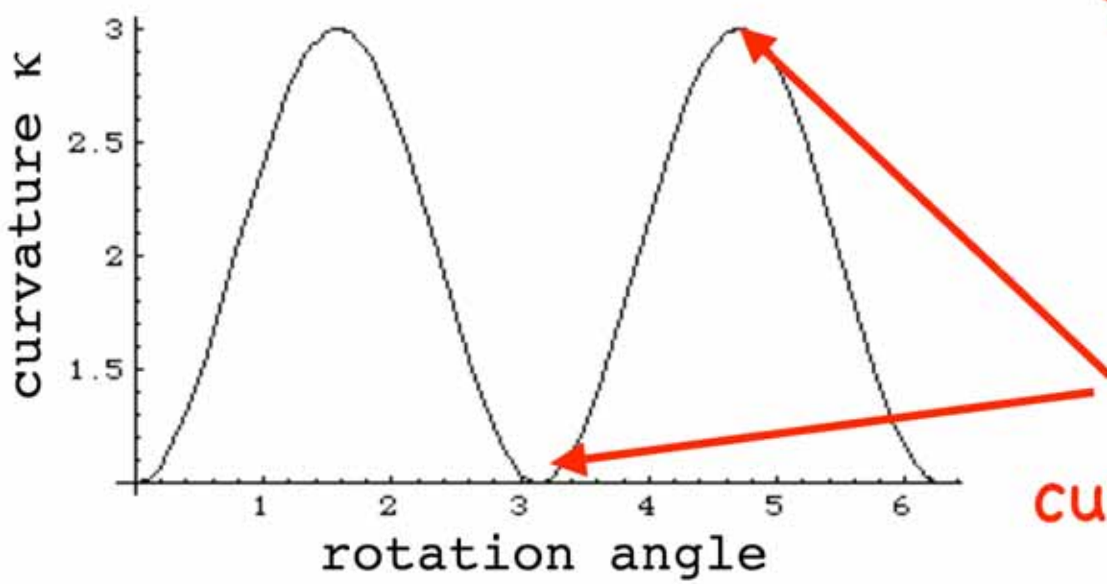

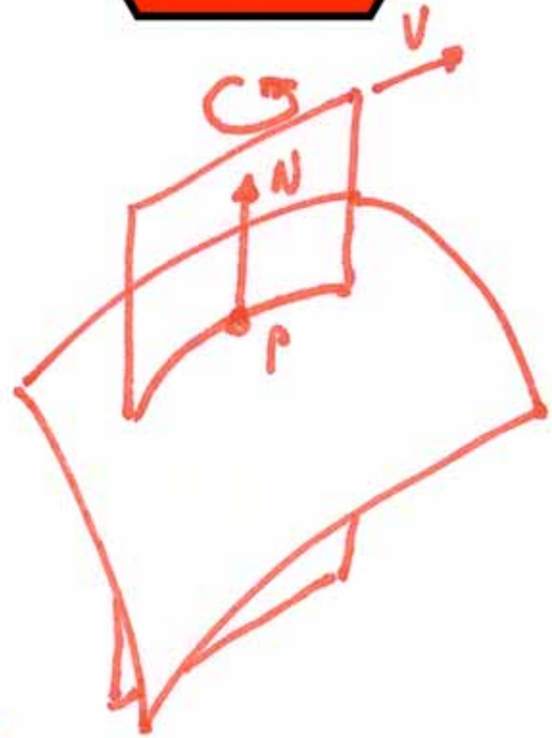

principal curvatures κ_1, κ_2

Principal Curvatures

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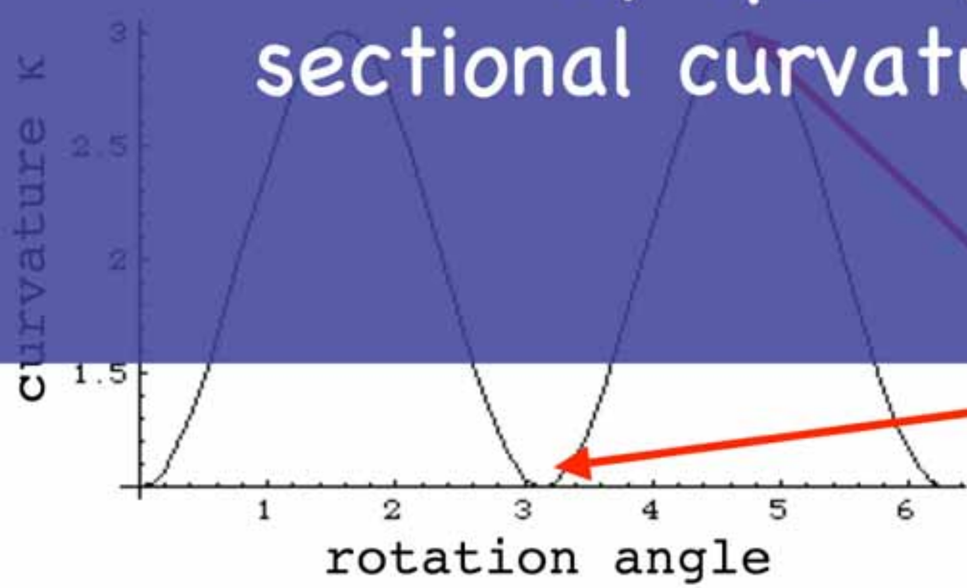
κ_1 and κ_2
fully specify all
sectional curvatures at P



principal curvatures κ_1, κ_2

Mean & Gaussian Curvature

Caution
smooth case only

Elementary symmetric functions of κ_1, κ_2

- Gaussian curvature $K = \kappa_1 \kappa_2$
- mean curvature $H = \kappa_1 + \kappa_2$

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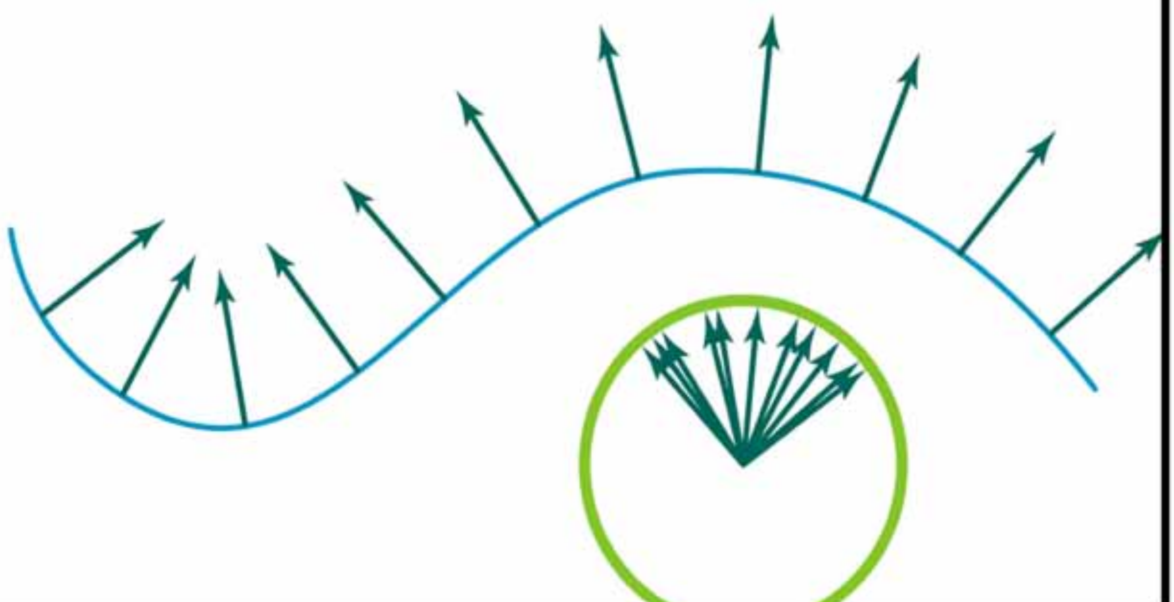
Solution: look for key properties of K and H

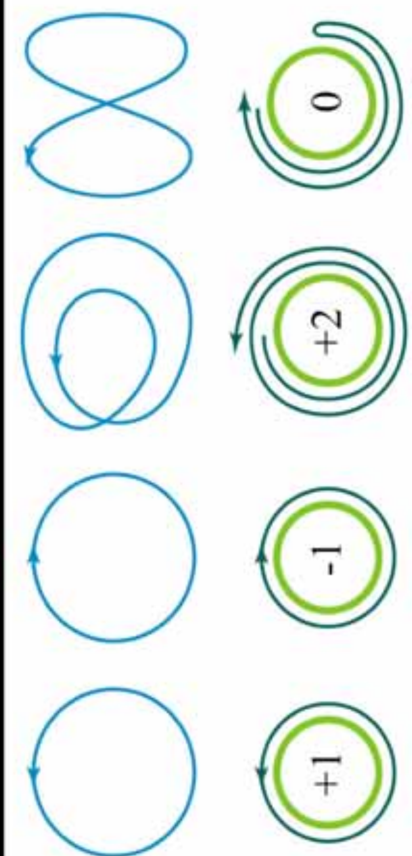
Gaussian Curvature

recall:

Gauß map, $\widehat{\mathrm{n}}(\mathrm{x})$

Point on curve maps to point on unit circle.





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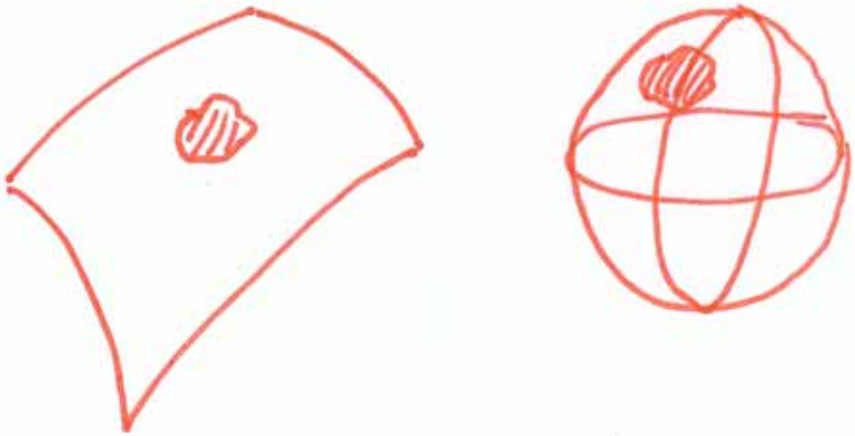
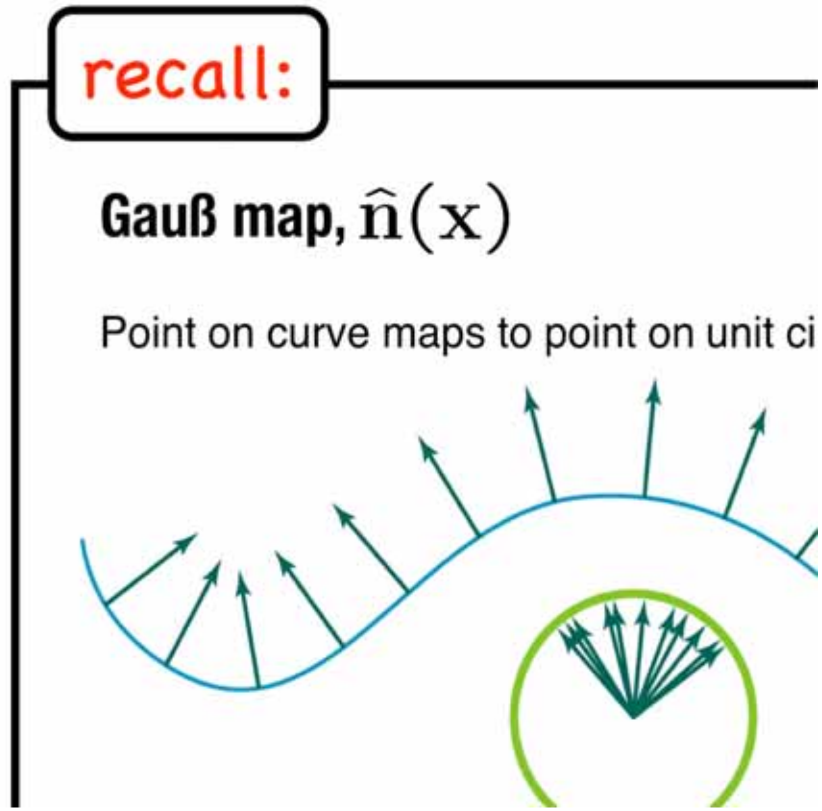


$$K_p = \lim_{A \to 0} \frac{A_G}{A}$$



Preserve Gauss-Bonnet Theorem

Notion of integrated Gauss curvature as area of region on unit sphere

- Gauss-Bonnet Theorem

Preserve Gauss-Bonnet Theorem

recall:

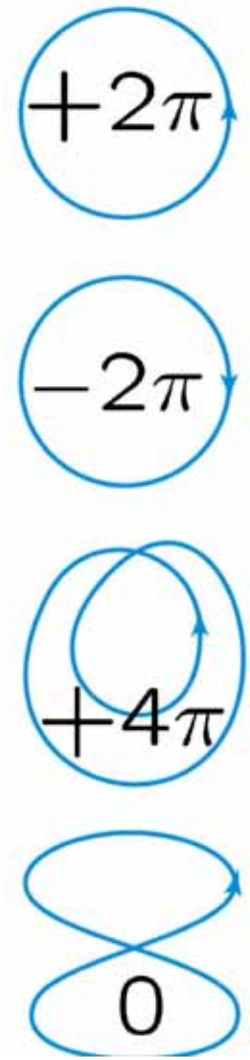
Notion
area

- Ga

Turning number theorem

$$\int_{\Omega} \kappa ds = 2\pi k$$

For a closed curve,
the integral of curvature is
an integer multiple of 2π.



Preserve Gauss-Bonnet Theorem

Notion of integrated Gauss curvature as area of region on unit sphere

- Gauss-Bonnet Theorem

$$2\pi\chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA$$

$2-2g$

for closed, oriented surface

$|f|-|e|+|v|$

for a simplicial complex

Gaussian Curvature

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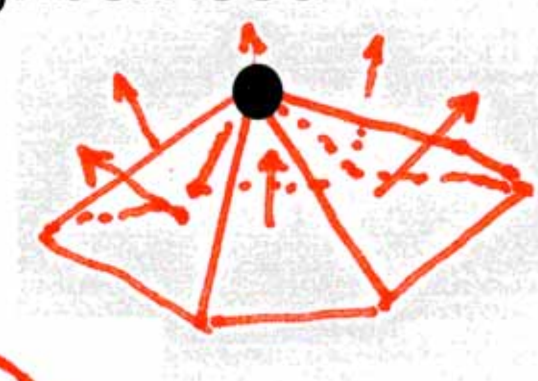

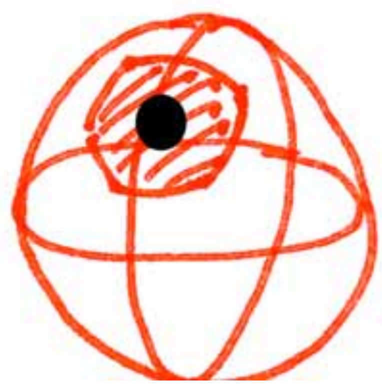

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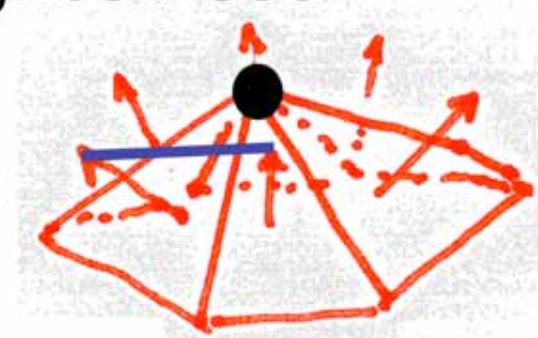

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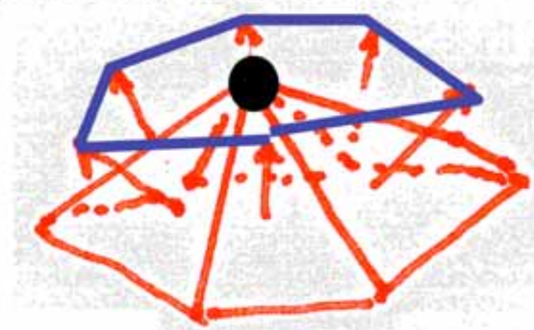

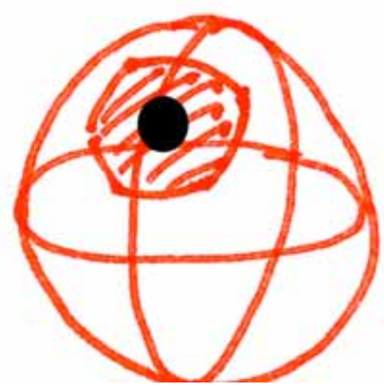

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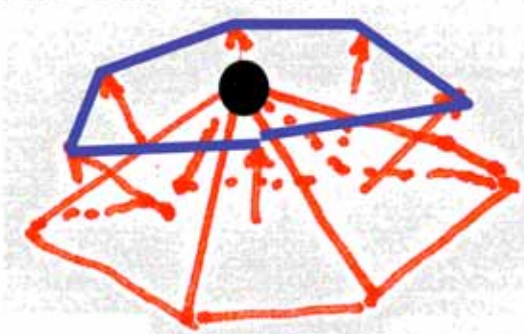
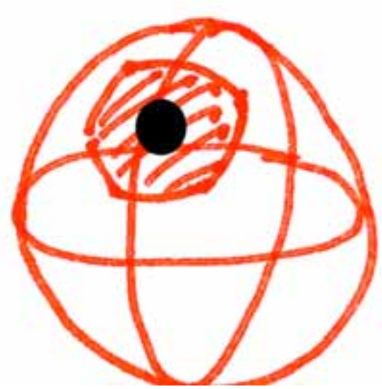
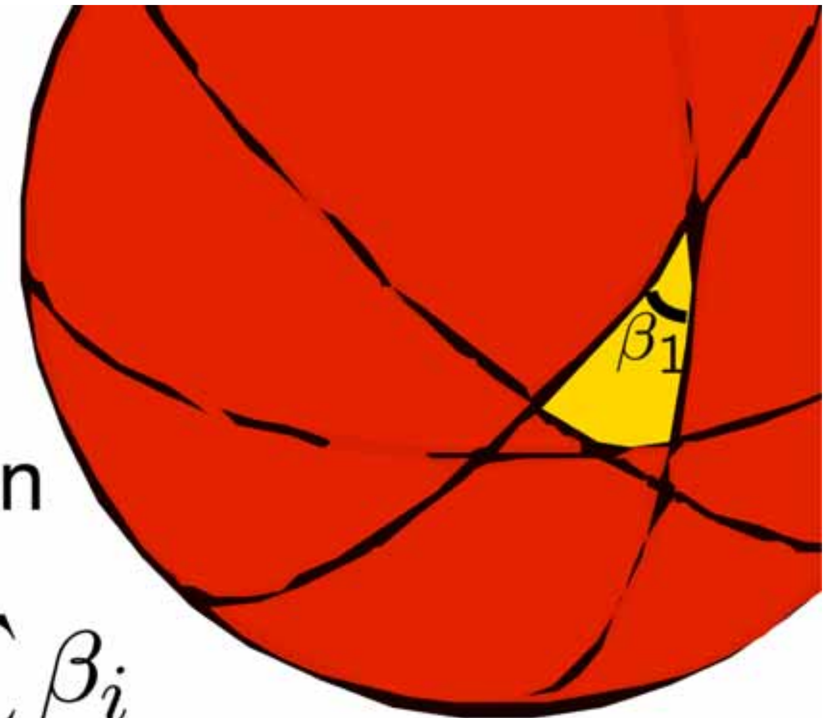


- our task:
  compute area of spherical polygon



Gaussian Curvature



Area of spherical polygon

$$A = (2-n)\pi + \sum_{i}^{n} \beta_i$$

Gaussian Curvature

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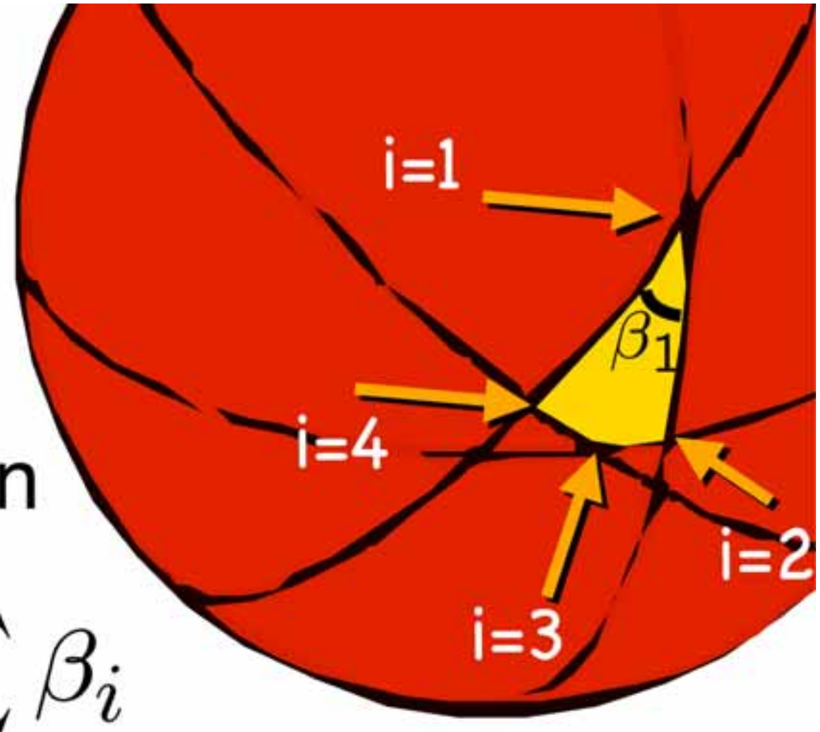

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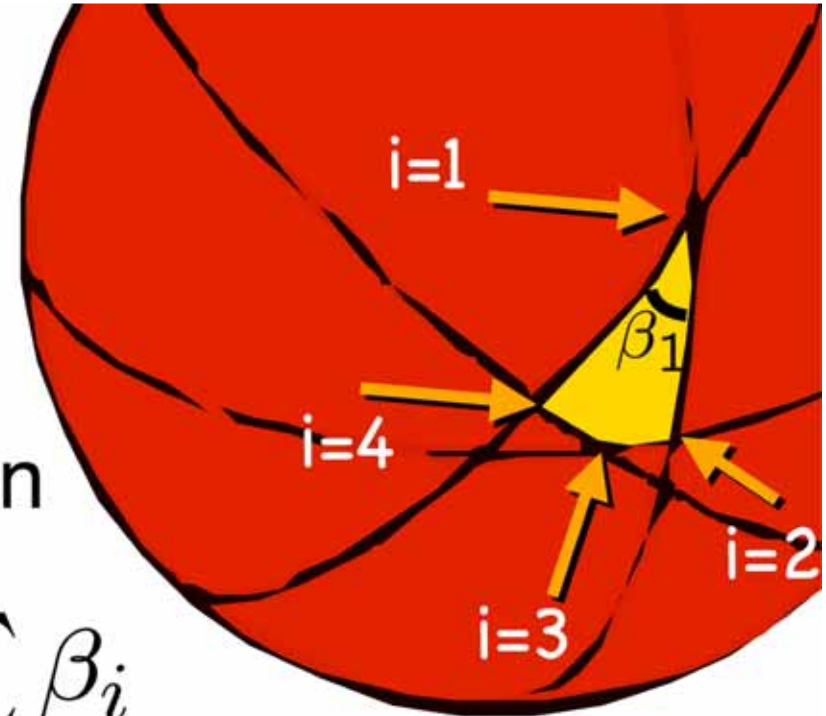

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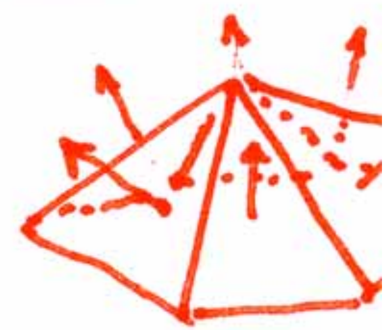
total Gauss curvature at vertex



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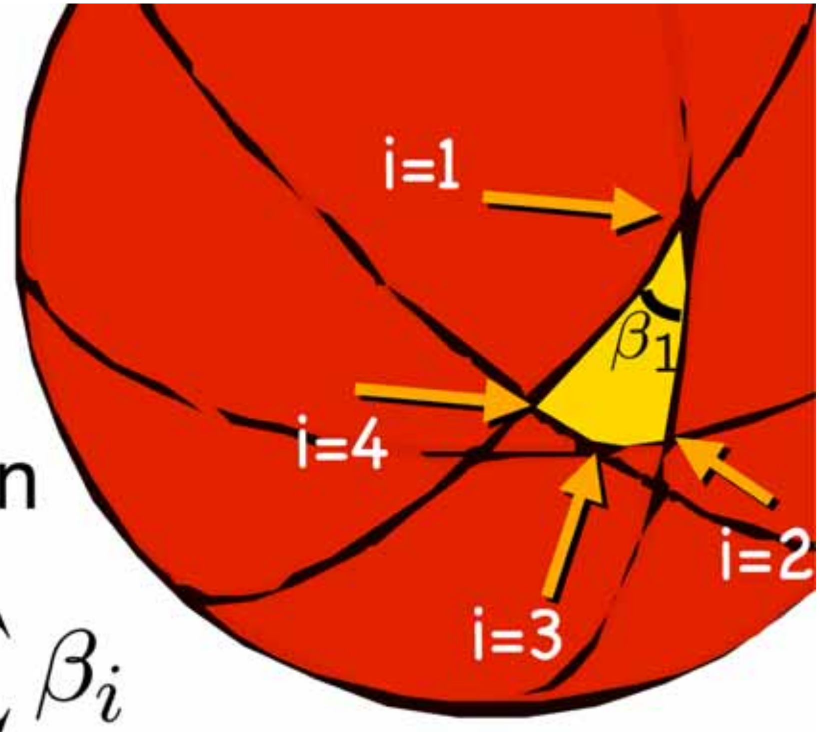

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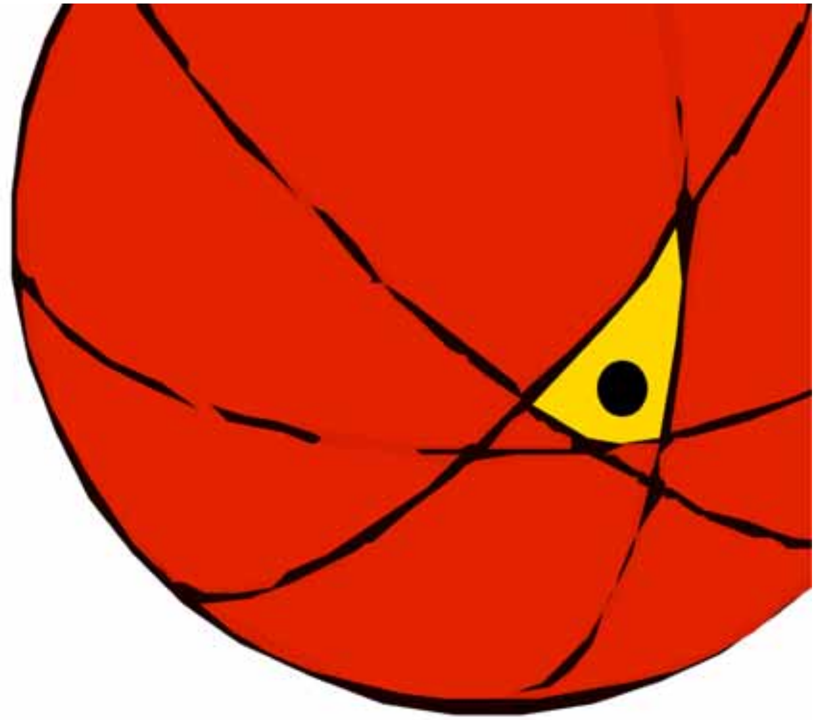
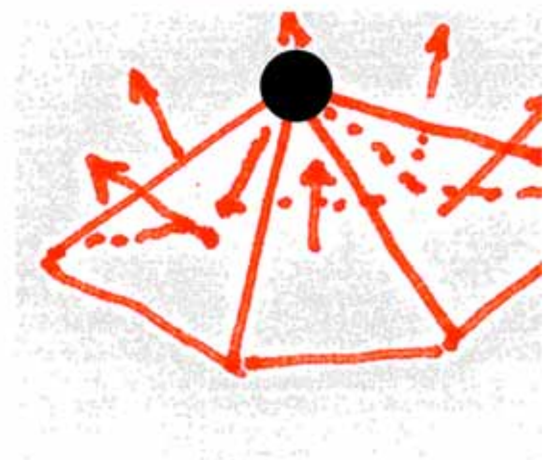
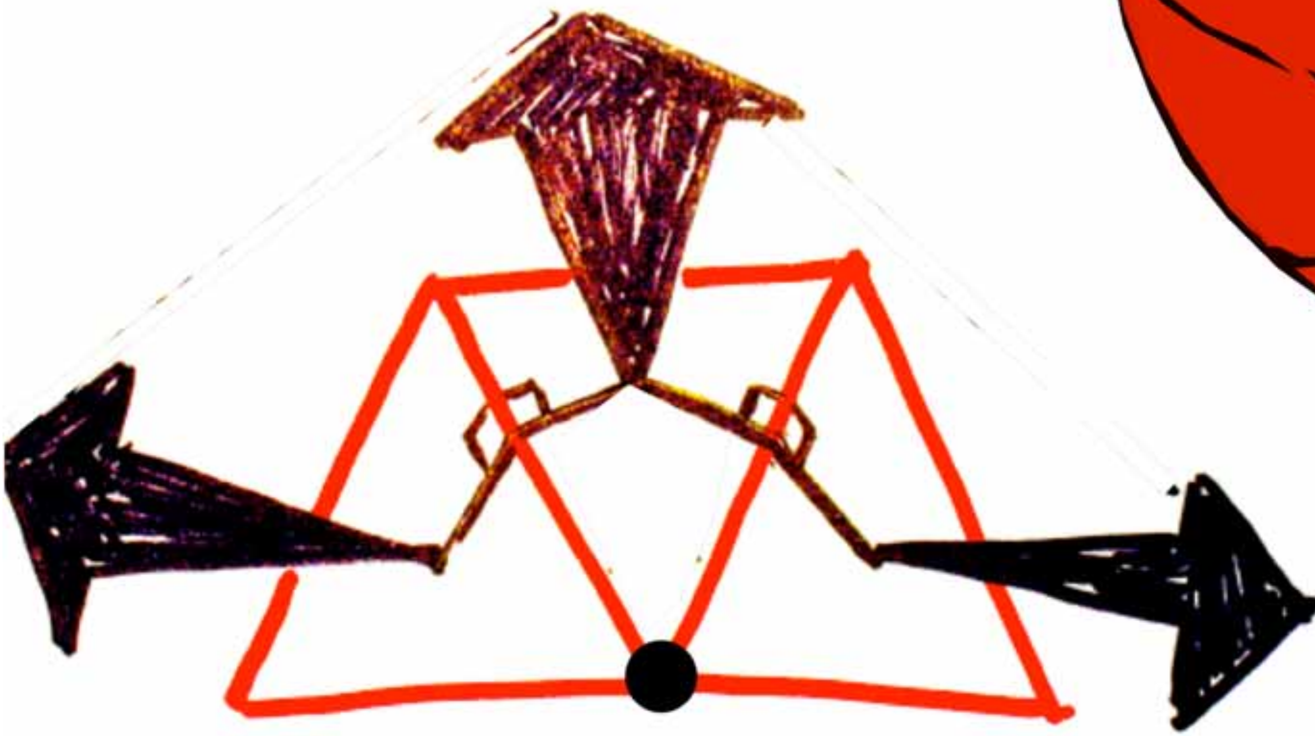
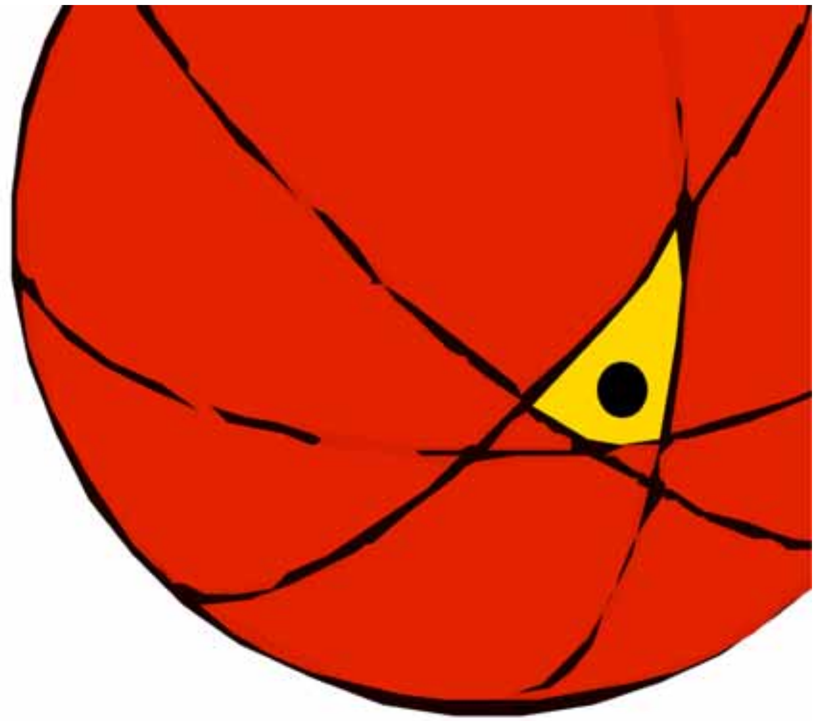
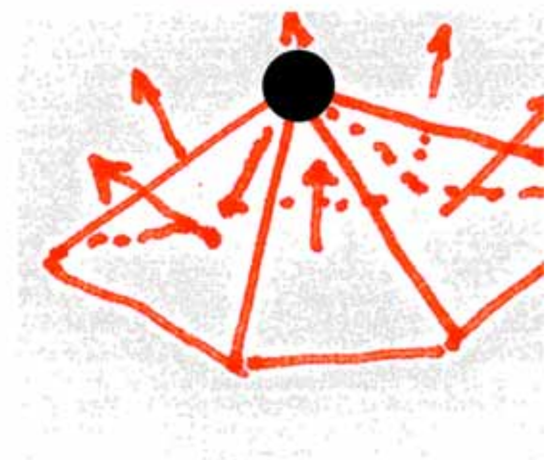
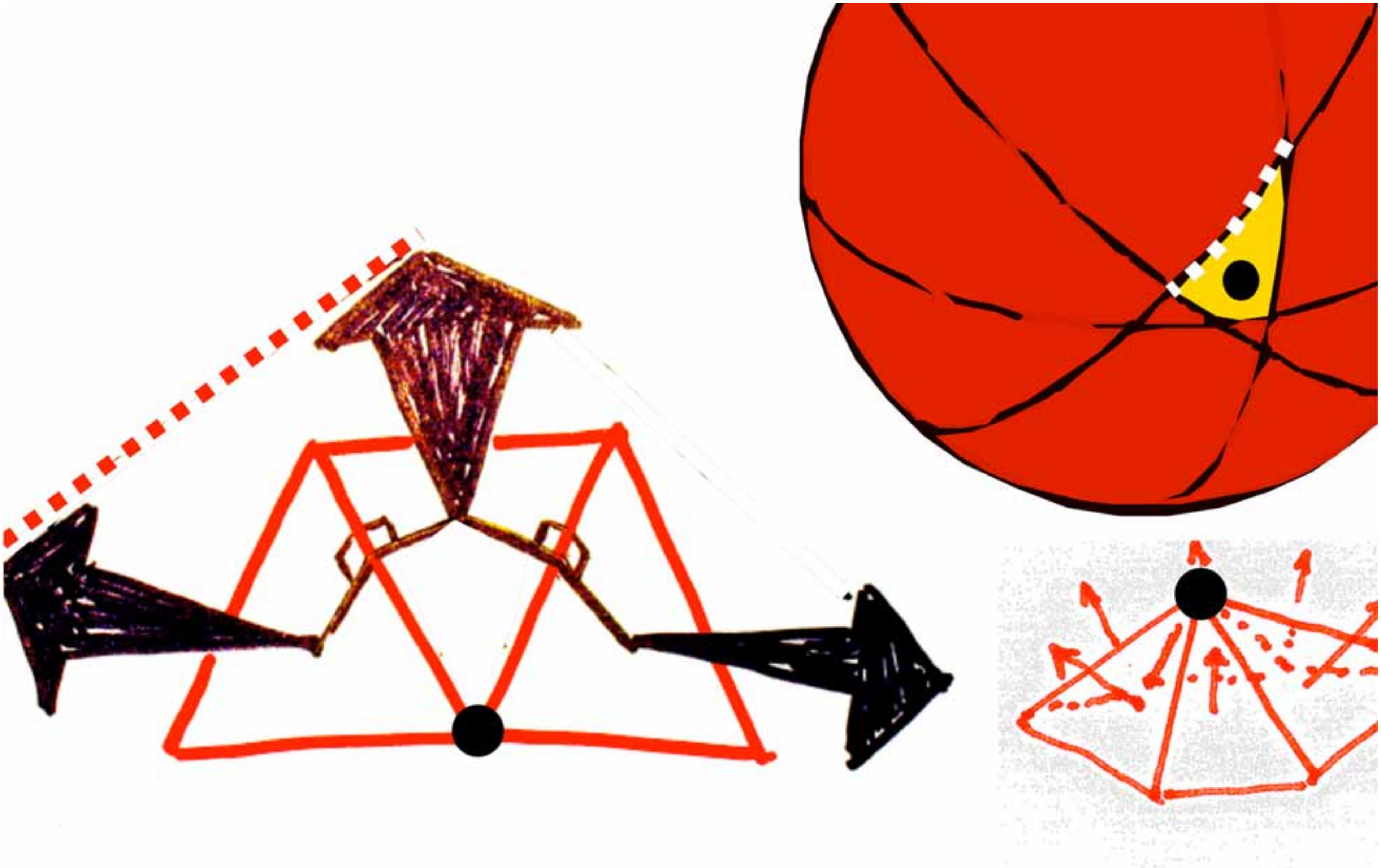
- where do I find β_i on my mesh?

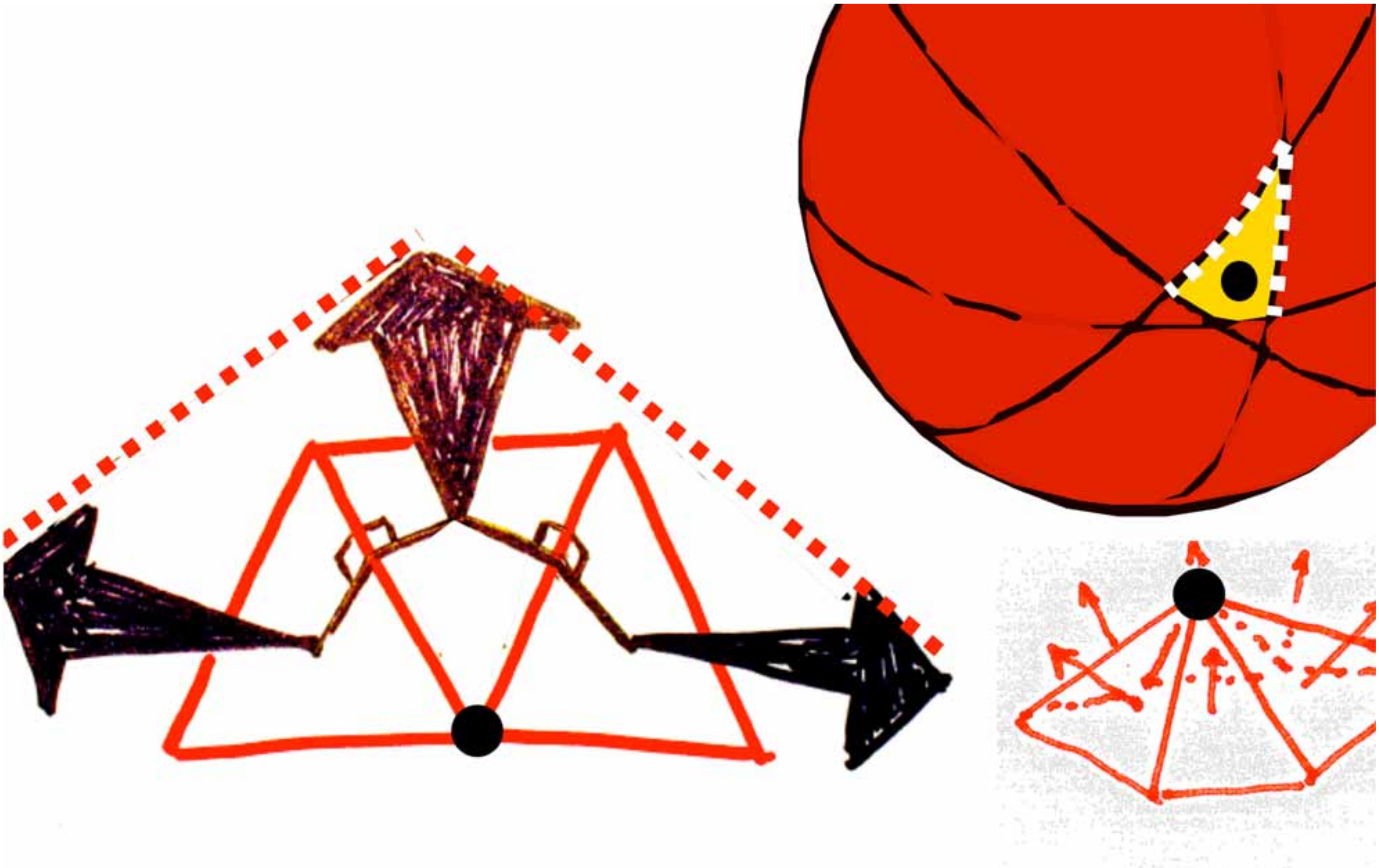



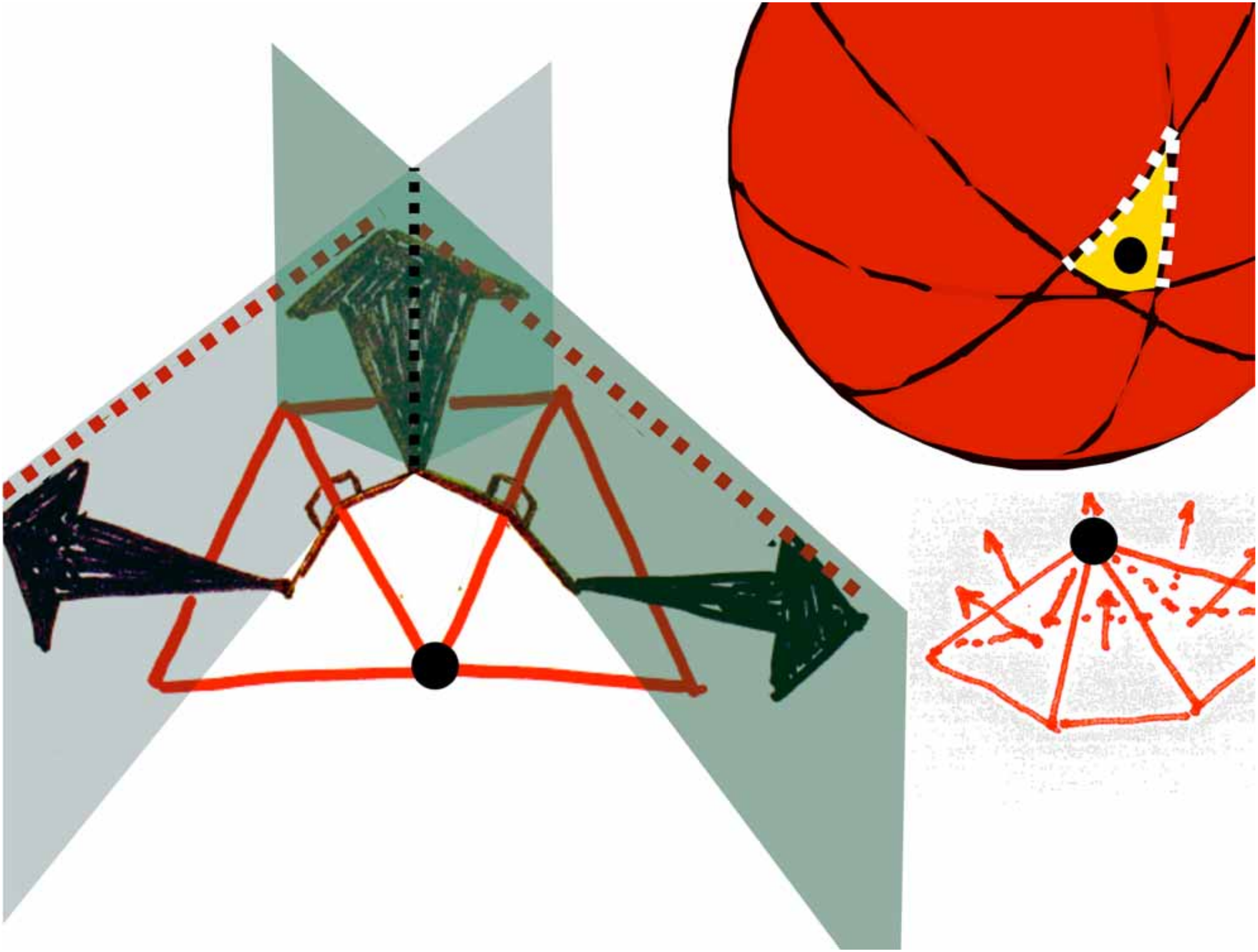



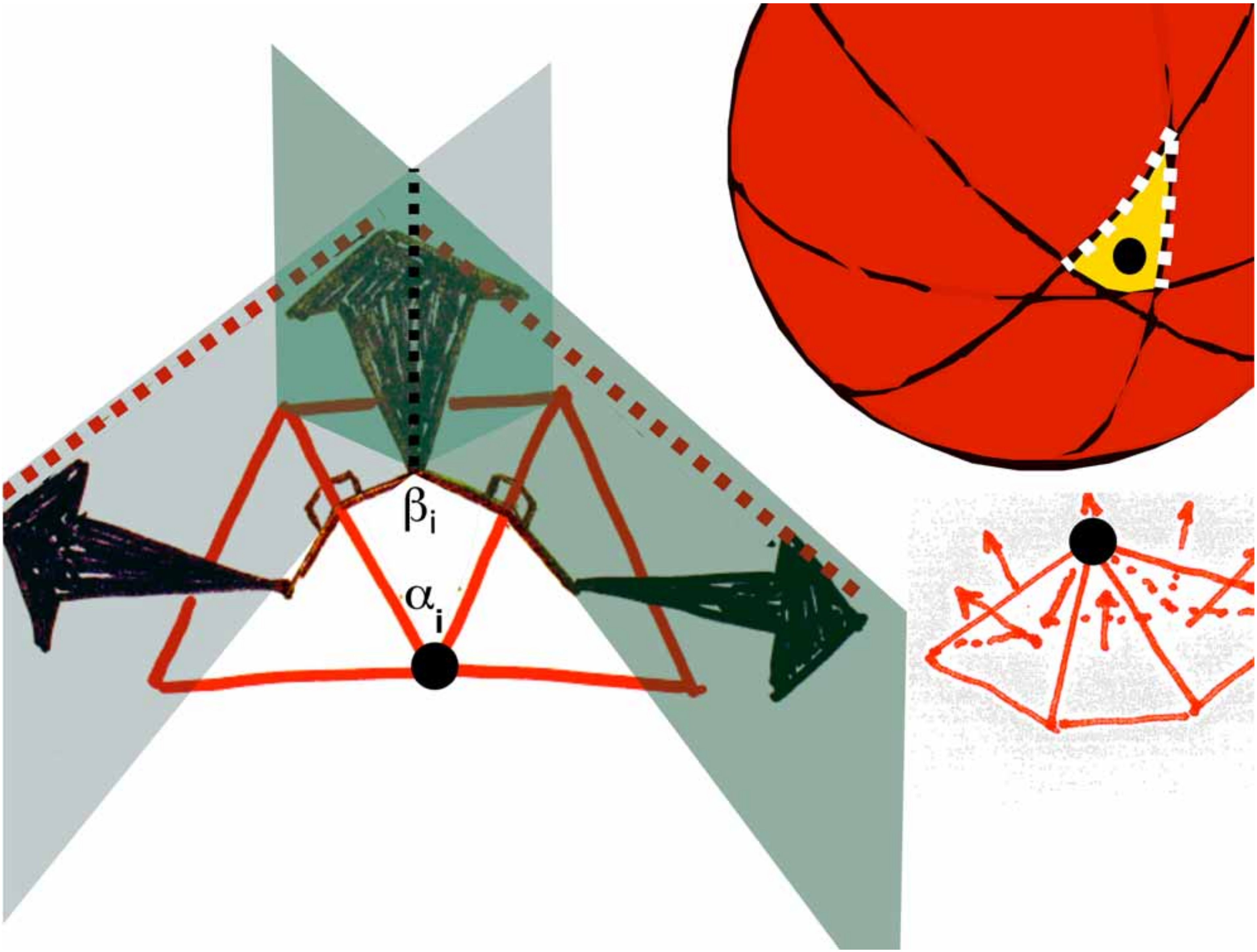




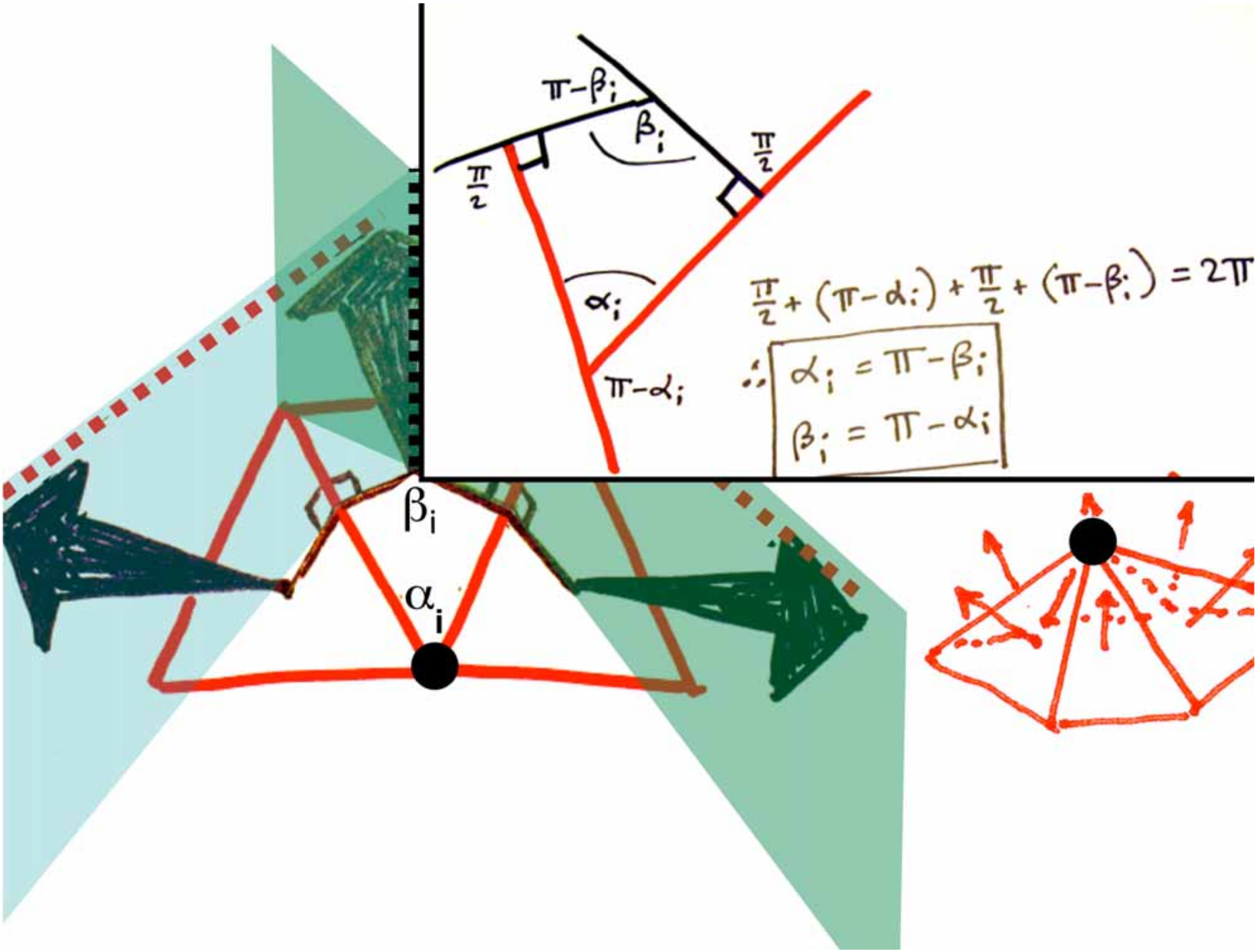








β_i
α_i

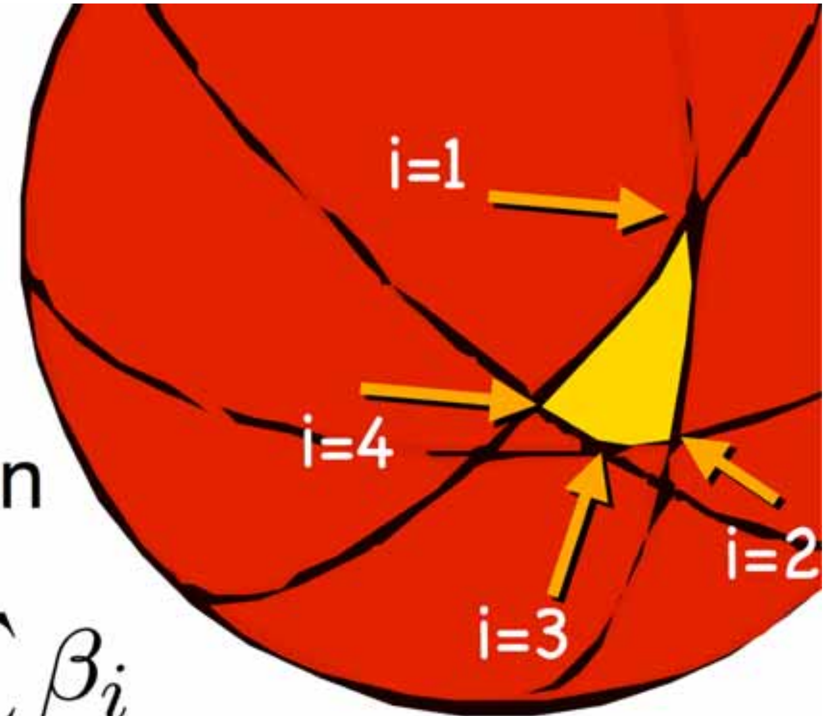


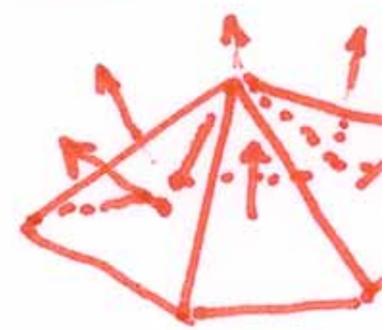
π−β_i
β_i
π/2
π/2
α_i
π−α_i
π/2 + (π−α_i) + π/2 + (π−β_i) = 2π
∴ α_i = π−β_i
β_i = π−α_i
β_i
α_i

Gaussian Curvature

Area of spherical polygon

$$A = (2-n)\pi + \sum_{i}^{n} \beta_i$$



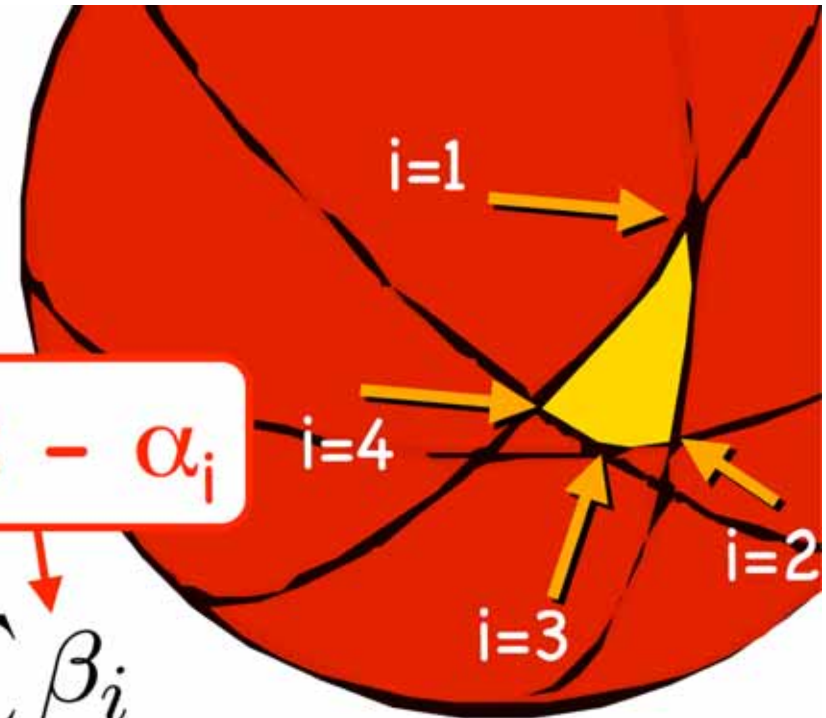


Gaussian Curvature

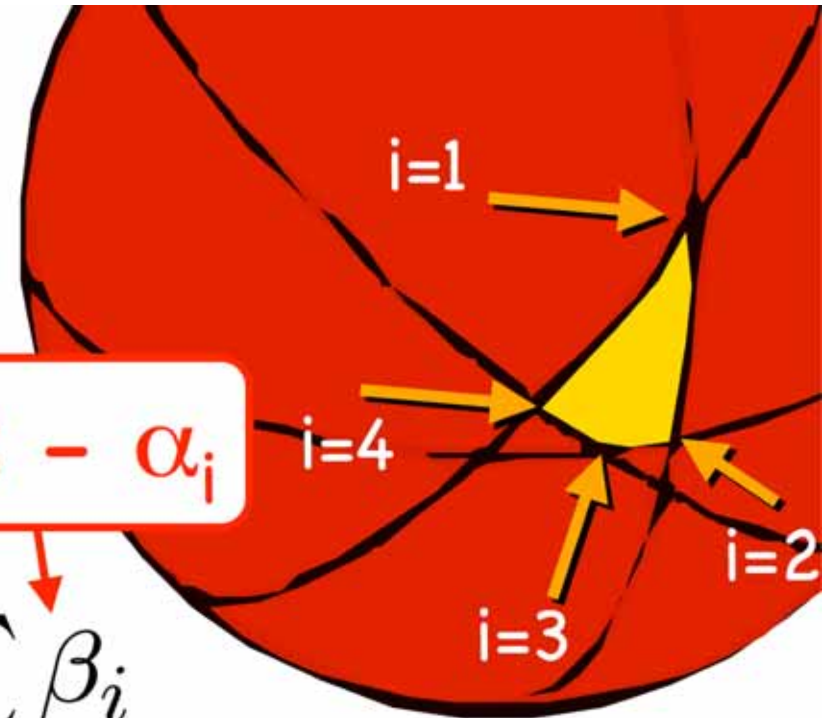
Area of spherical poly

$\pi - \alpha_i$

$$A = (2-n)\pi + \sum_{i}^{n} \beta_i$$



Gaussian Curvature



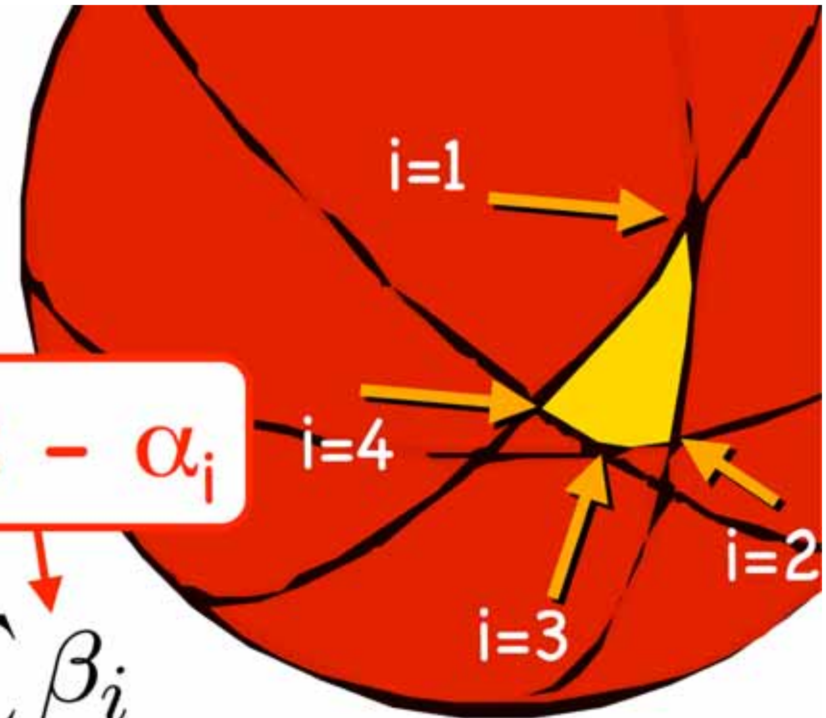
Area of spherical poly

$\pi - \alpha_i$

$$A = (2-n)\pi + \sum_{i}^{n} \beta_i$$

$$A = (2-n)\pi + n\pi - \sum_{i}^{n} \alpha_i$$

Gaussian Curvature



Area of spherical poly

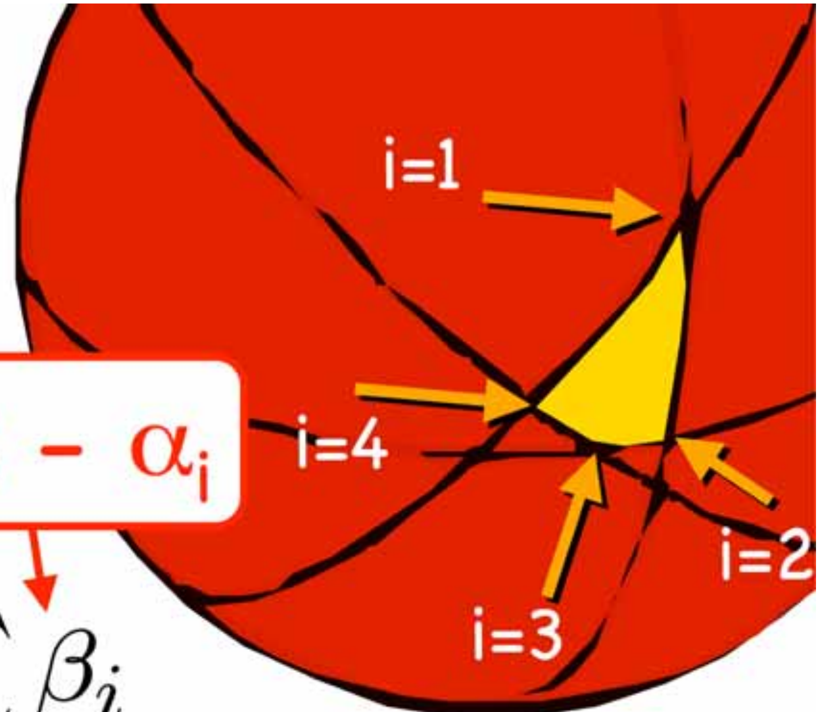
$$A = (2-n)\pi + \sum_{i}^{n} \beta_i$$

$$A = (2 \cancel{- n)\pi + n\pi} - \sum_{i}^{n} \alpha_i$$

$$A = 2\pi - \sum_{i}^{n} \alpha_i$$

Gaussian Curvature

Area of spherical poly



$\pi - \alpha_i$

$$A = (2-n)\pi + \sum_{i}^{n} \beta_i$$

$$A = (2-n)\pi + n\pi - \sum_{i}^{n} \alpha_i$$

$$A = 2\pi - \sum_{i}^{n} \alpha_i$$

total Gauss curvature at vertex

Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

- Gauss-Bonnet

Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

- Gauss-Bonnet

$$2\pi\chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA$$

|f|-|e|+|v|

Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

- Gauss-Bonnet

$$2\pi\chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA$$

|f|-|e|+|v|

- discrete

Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

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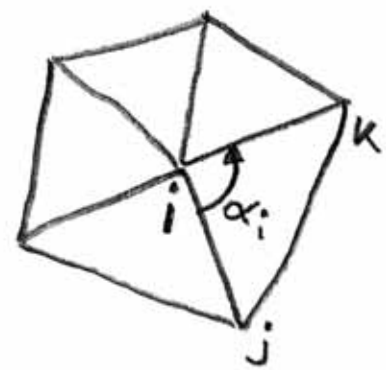
$$2\pi\chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA$$

|f|-|e|+|v|

- discrete

$$K_i = 2\pi - \sum_{t_{ijk}} \alpha_{jk}$$



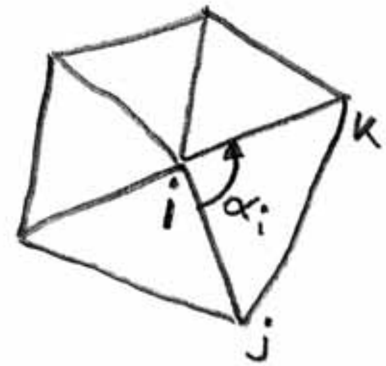
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

- Gauss-Bonnet

$$2\pi\chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA$$

|f|-|e|+|v|



- discrete

$$K_i = 2\pi - \sum_{t_{ijk}} \alpha_{jk}$$

$$\sum_i K_i = 2\pi(V - F/2) = 2\pi(F - 3F/2 + V) = 2\pi\chi$$

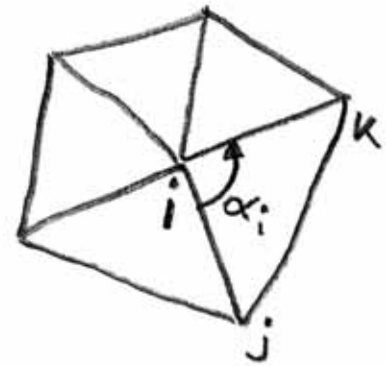
Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

- Gauss-Bonnet

$$2\pi\chi = \int_S \kappa_1 \kappa_2 \, dA = \int_S K \, dA$$

|f|-|e|+|v|



- discrete

$$K_i = 2\pi - \sum_{t_{ijk}} \alpha_{jk}$$

E

$$\sum_i K_i = 2\pi(V - F/2) = 2\pi(F - 3F/2 + V) = 2\pi\chi$$

face angles
sum to π

Discrete Gauss-Bonnet

Gauss-Bonnet satisfied *exactly*

- Gauss-Bonnet

$$2\pi\chi = \int_S \kappa_1\kappa_2 \, dA = \int_S K \, dA$$

|f|-|e|+|v|

- discrete

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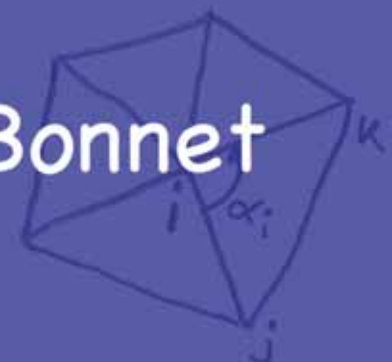


in discrete setting,
it's easy to prove Gauss-Bonnet

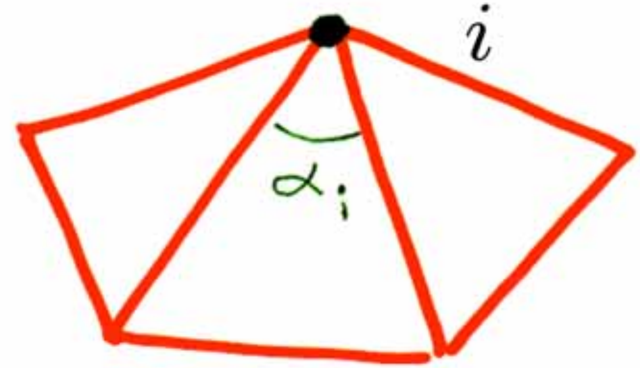
E

$$\sum_i K_i = 2\pi(V - F/2) = 2\pi(F - 3F/2 + V) = 2\pi\chi$$

face angles
sum to π

Gaussian Curvature

$$A = 2\pi - \sum_{i}^{n} \alpha_i$$



Intrinsic curvature

- sees only in-plane angles
- does not depend on embedding

Discrete setting

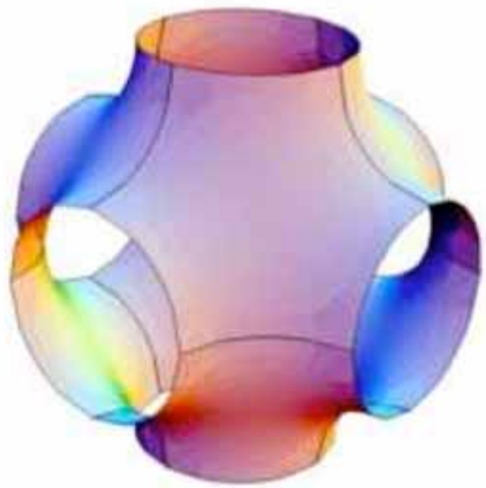
- only pedestrian calculations required to evaluated, and to prove Gauss-Bonnet
- associated to vertex neighborhood

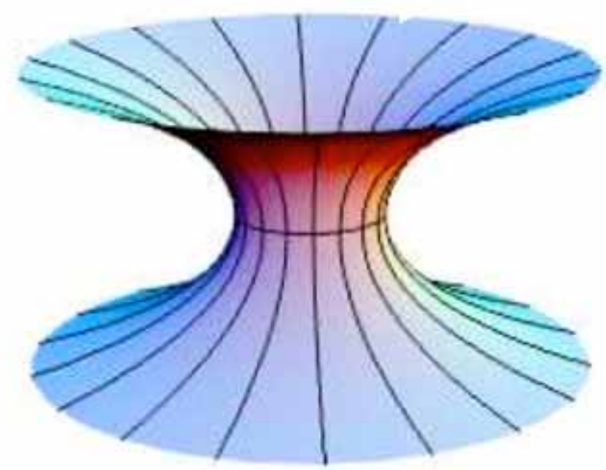
think *total* Gauss curvature near vertex

Mean Curvature $(\kappa_1 + \kappa_2)$

Variational structure of mean curvature

- surfaces which minimize area
  - soap bubbles
- at any given point:
  - $k_1 = -k_2$
  - $H = 0$
  - $\mathbf{H} = H\mathbf{N} = 0$





Mean Curvature $(\kappa_1 + \kappa_2)$

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- surfaces which minimize area
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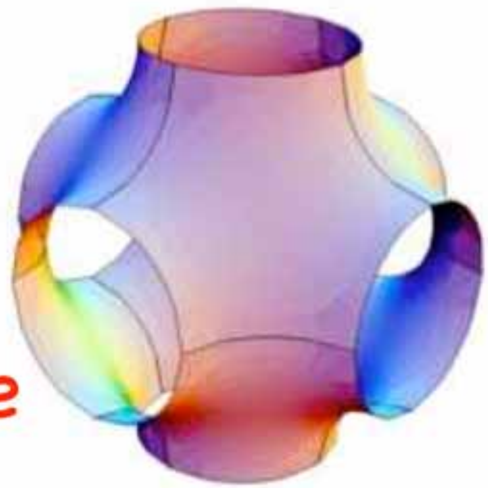
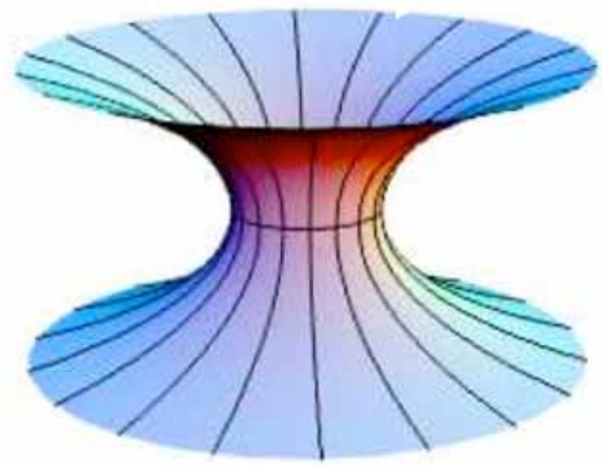
mean curvature <u>vector</u>





Mean Curvature $(\kappa_1 + \kappa_2)$

Variational structure of mean curvature

- surfaces which minimize area
  - soap bubbles
- at any given point:
  - $\kappa_1 = -\kappa_2$
  - H = 0
  - **H**=H**N**=0

area is (locally) minimum

iff

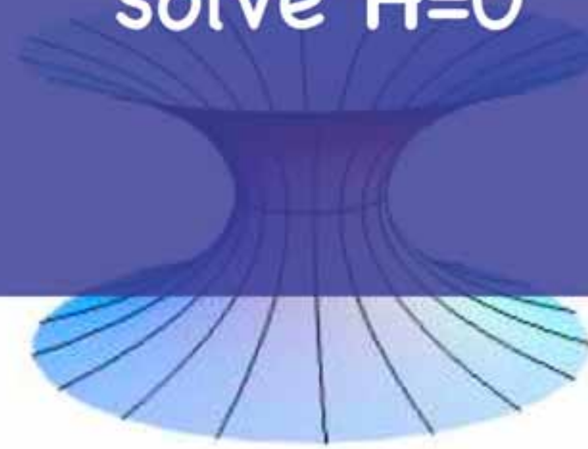
mean curvature is zero

minimize A

≡

solve H=0

mean curvature <u>vector</u>

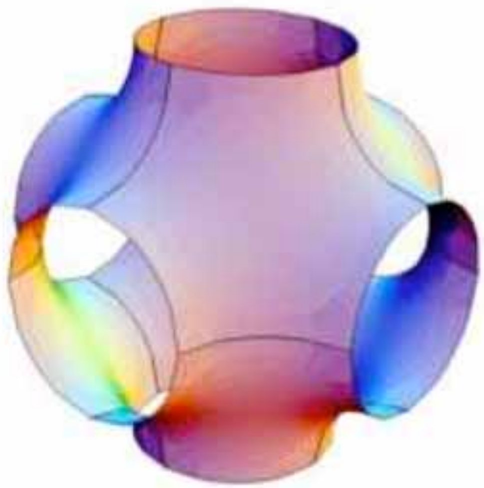


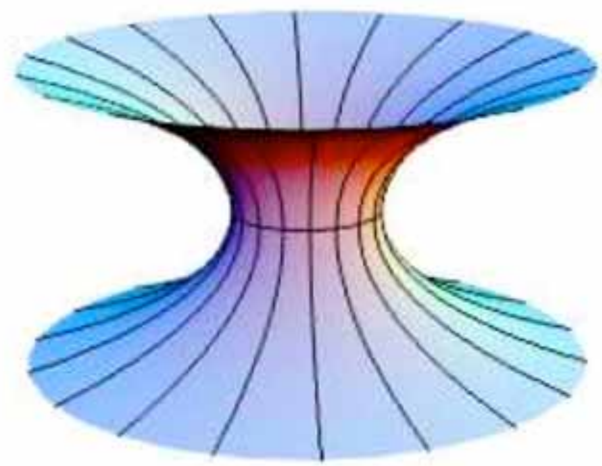
Mean Curvature Vector

$$\vec{H} = \text{grad area}$$

Calculus of Variations

- stationary area $\Leftrightarrow$ grad area = $\vec{\mathbf{H}}$ = 0





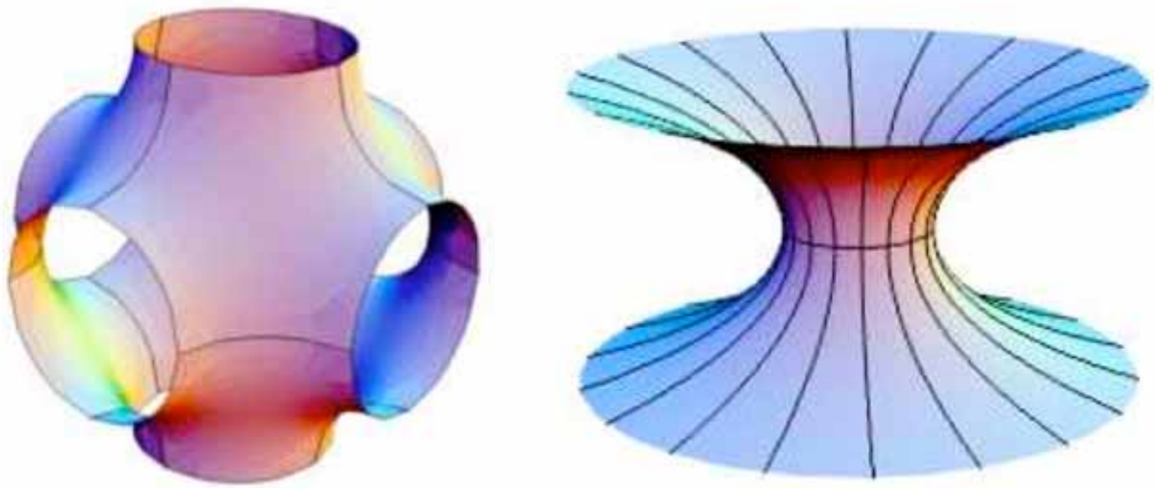
Mean Curvature Vector

$\boxed{\vec{H} = \text{grad area}}$

some prefer
$H = -\text{grad area}$

Calculus of Variations

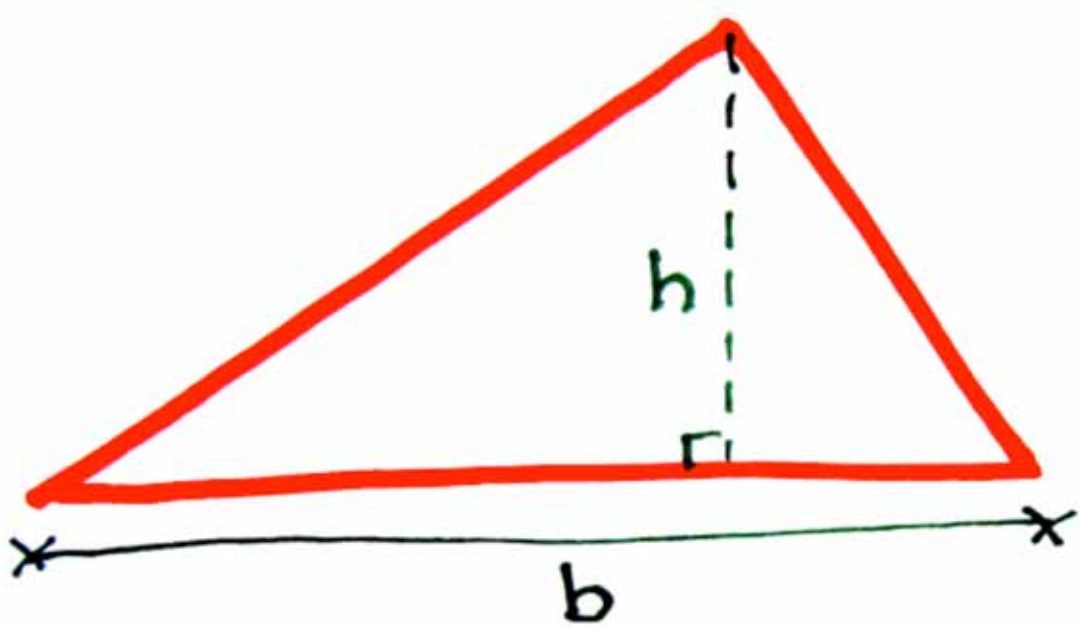
- stationary area $\Leftrightarrow$ grad area = $\vec{H} = 0$



Mean Curvature Vector

$$\vec{H} = \text{grad area}$$

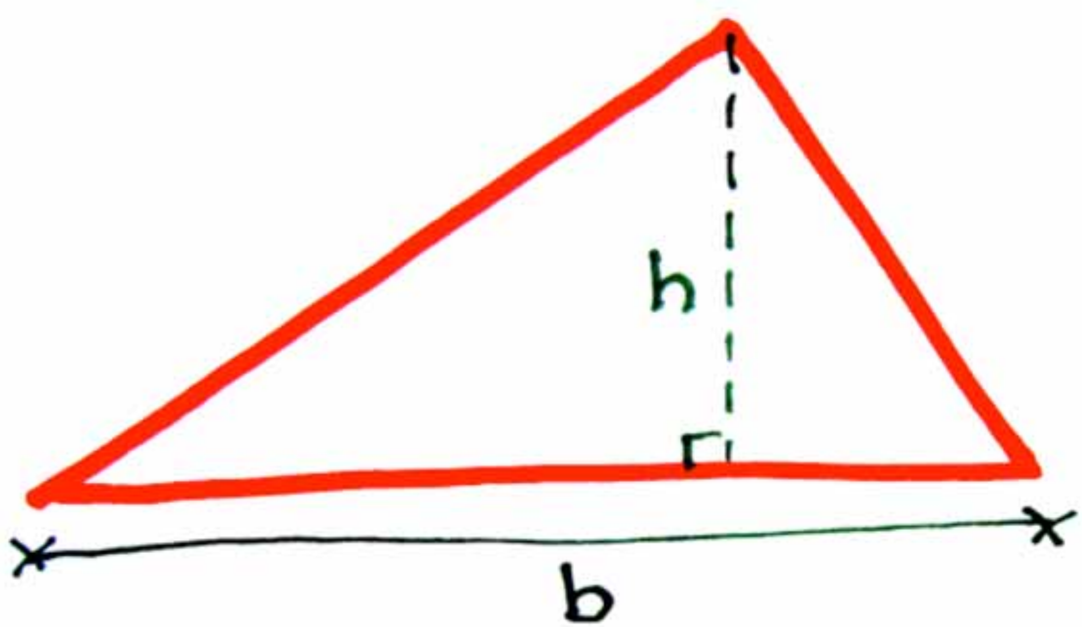
area = bh



Mean Curvature Vector

$$\boxed{\vec{H} = \text{grad area}}$$

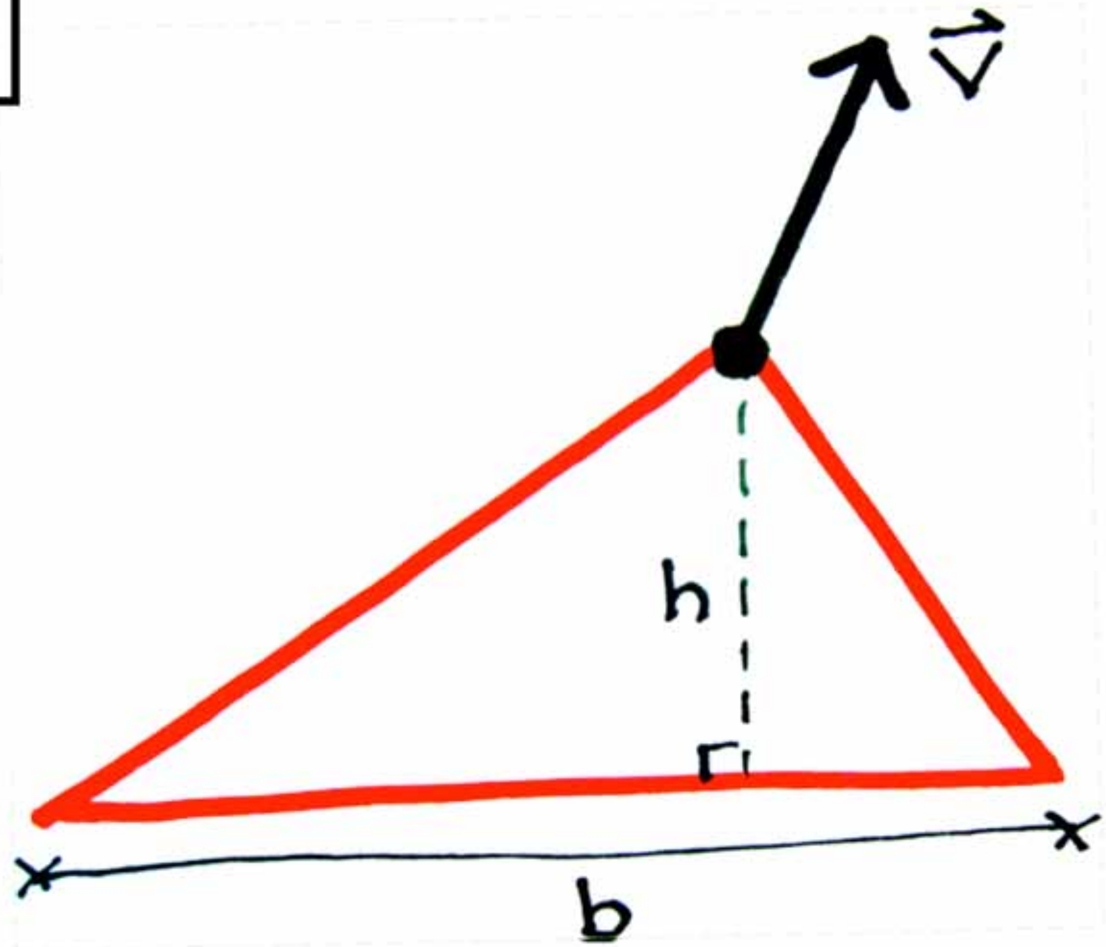
$2\times$ area = b h



Mean Curvature Vector

$\vec{H}$ = grad area

2x area = bh

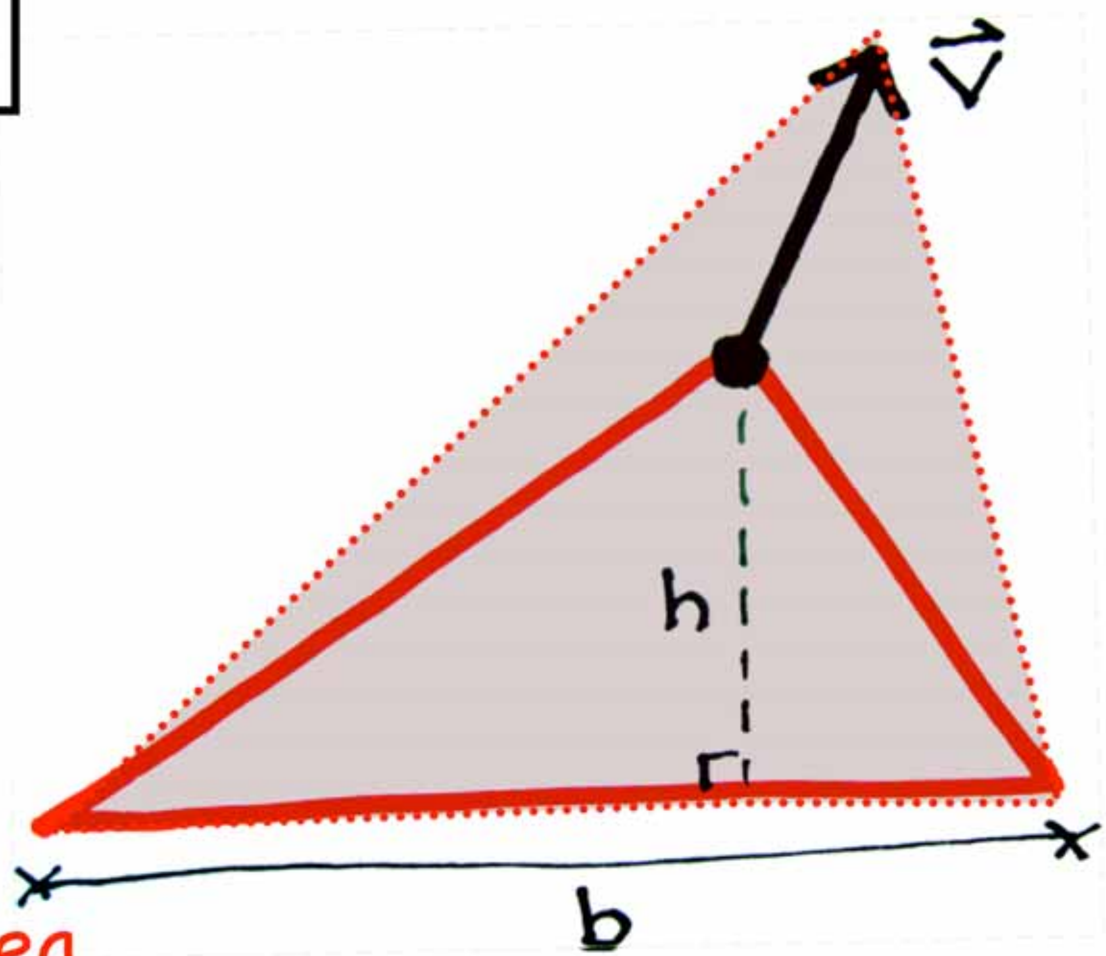


Mean Curvature Vector

$\vec{H}$ = grad area

2x area = bh

(grad area) v = Δarea



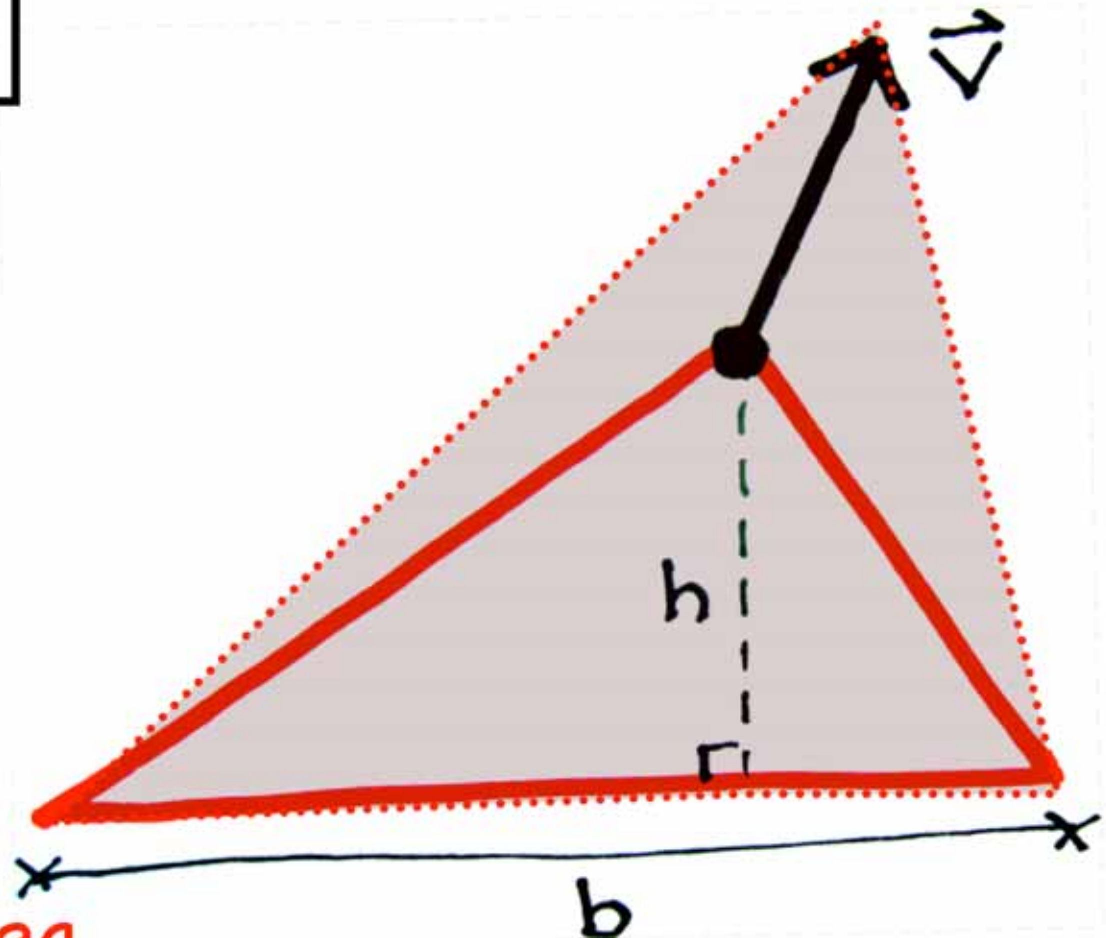
Mean Curvature Vector

$\vec{H}$ = grad area

2× area = bh

Δarea =

(grad area) v = Δarea

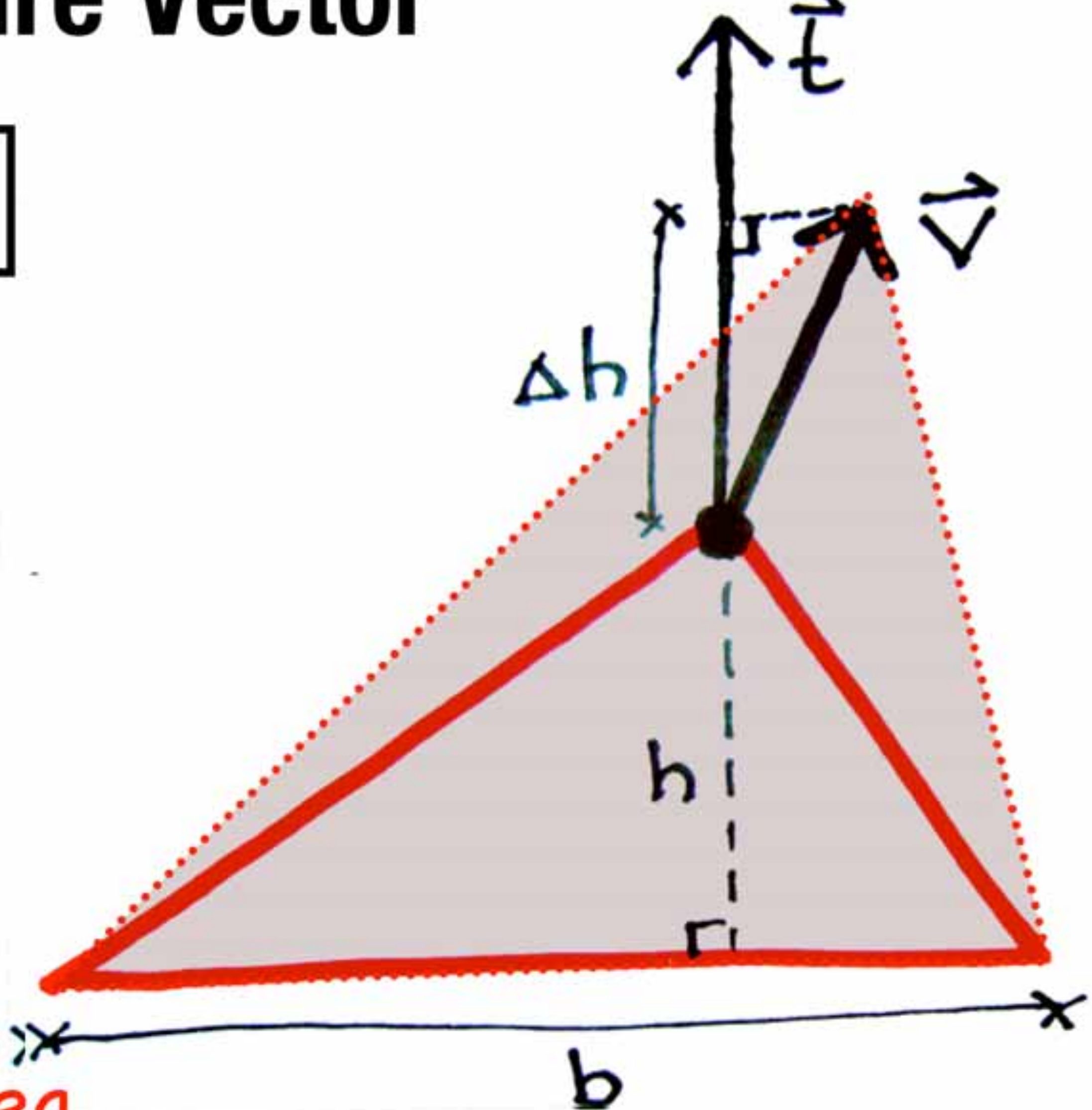


Mean Curvature Vector

$$\vec{H} = \text{grad area}$$

$$2 \times \text{area} = b h$$

$$\Delta \text{area} = b \, \Delta h$$



(grad area) v = Δarea

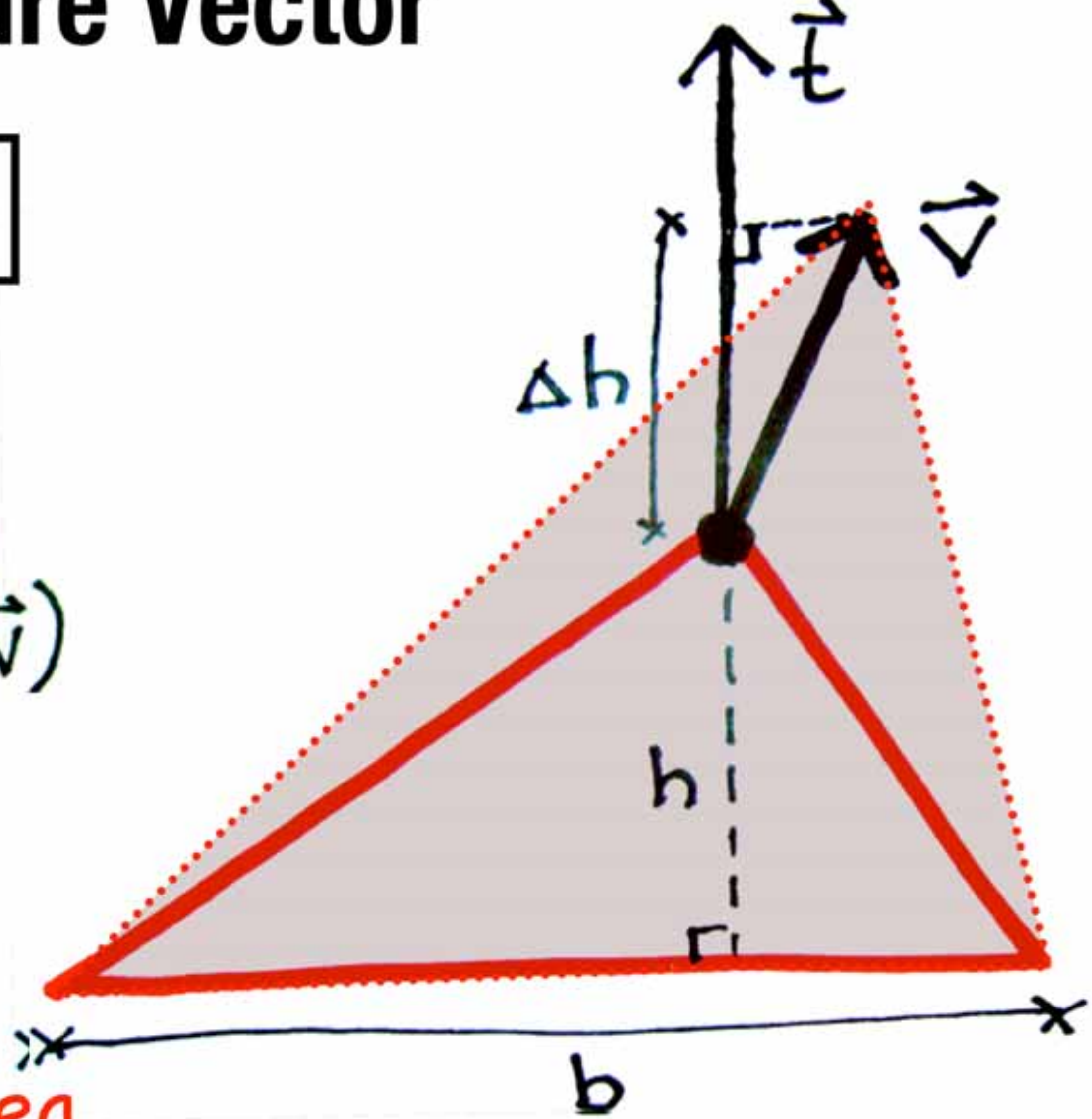
Mean Curvature Vector

$\vec{H}$ = grad area

$2 \times \text{area} = b\,h$

$\Delta\text{area} = b\,\Delta h$

$= b(\hat{t} \cdot \vec{v})$



(grad area) v = Δarea

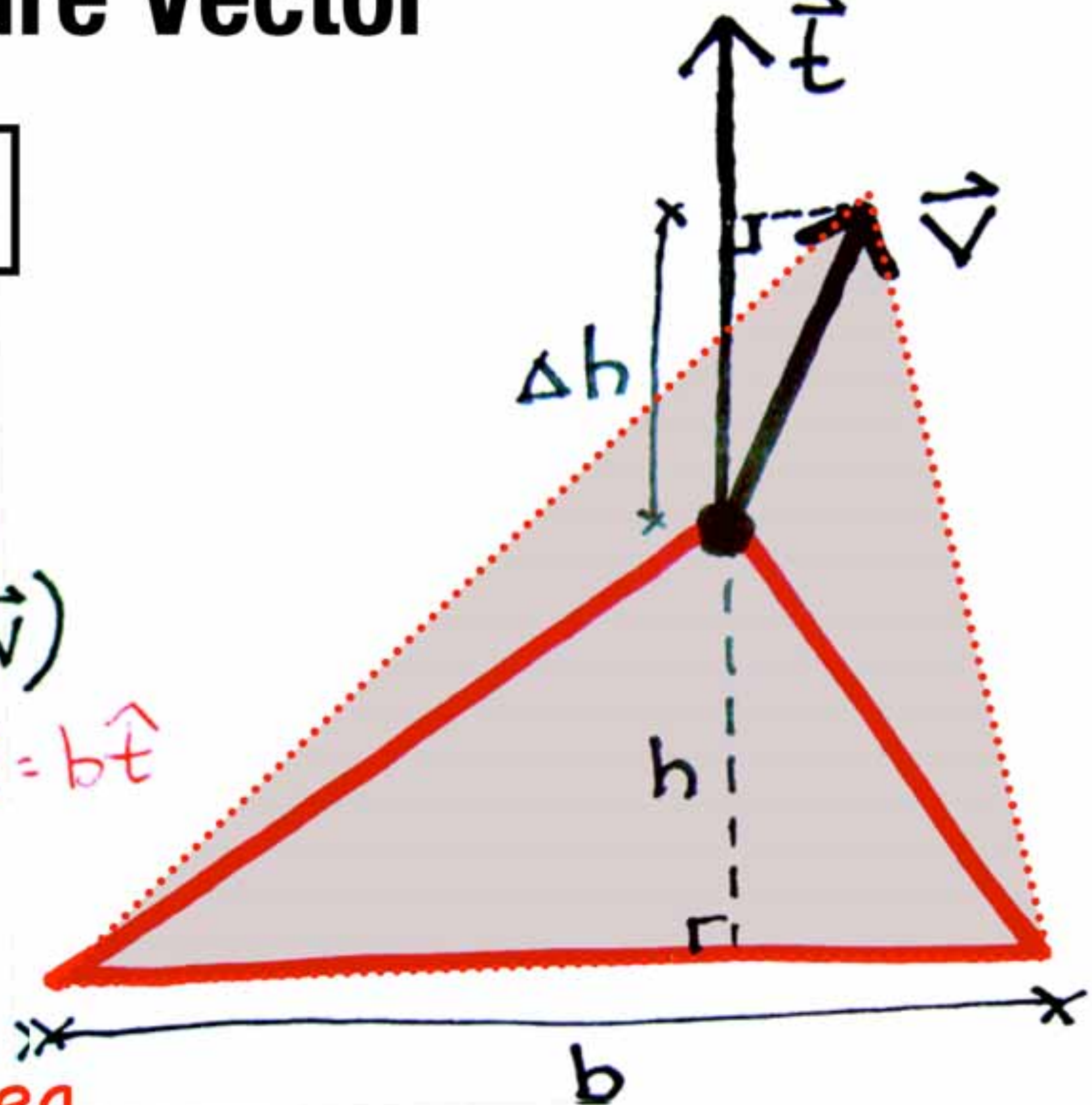
Mean Curvature Vector

$\vec{H}$ = grad area

$2\times \text{area} = bh$

$\Delta\text{area} = b\,\Delta h$

$= \underbrace{b}_{\text{let } \vec{t} = b\hat{t}}(\hat{t}\cdot\vec{v})$



(grad area) v = Δarea

Mean Curvature Vector

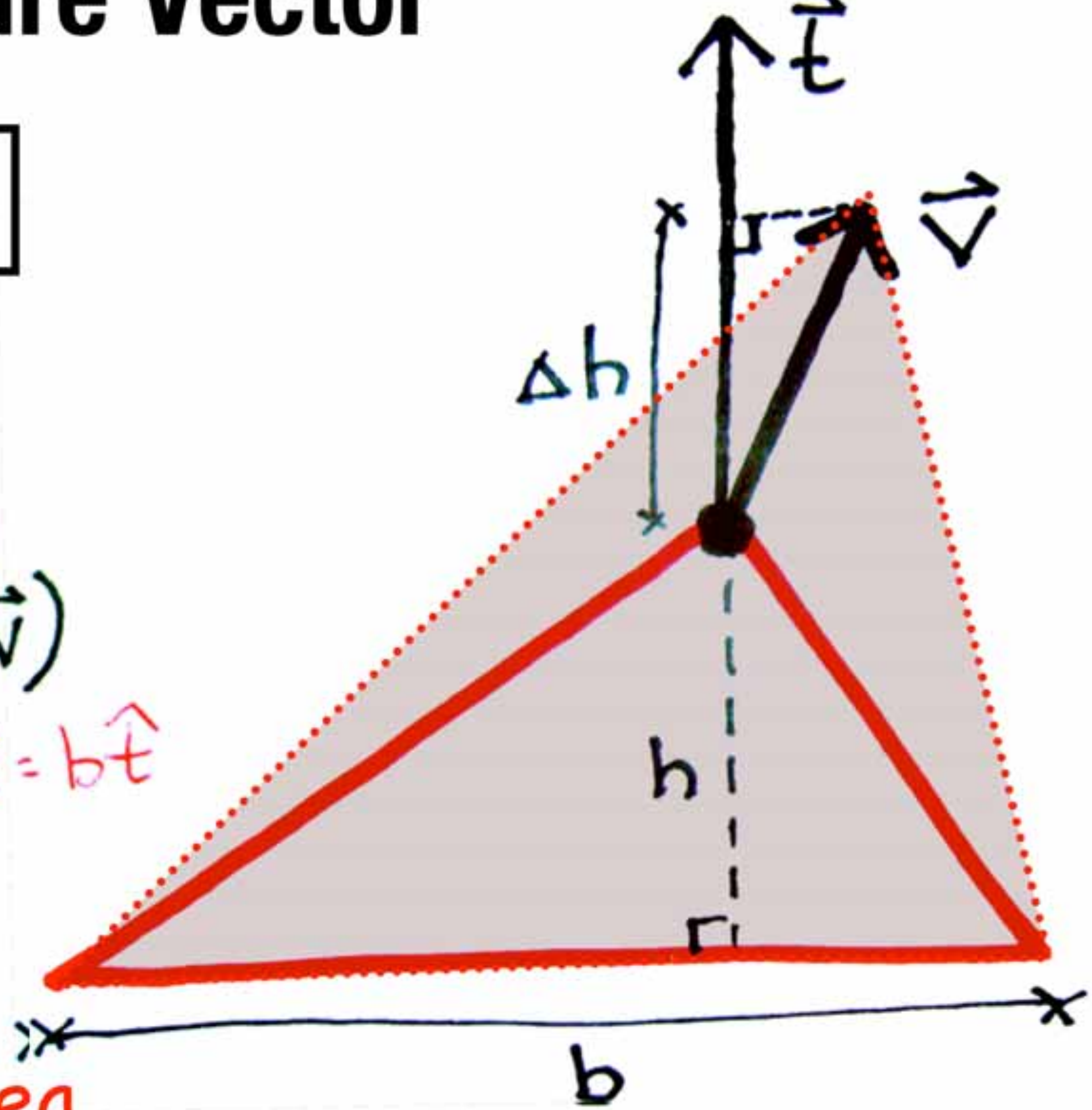
$$\vec{H} = \text{grad area}$$

$$2 \times \text{area} = b h$$

$$\Delta\text{area} = b\,\Delta h$$

$$= \underbrace{b\,(\hat{t}}_{\text{let } \vec{t} = b\hat{t}} \cdot \vec{v})$$

$$= \vec{b} \cdot \vec{v}$$



(grad area) v = Δarea

Mean Curvature Vector

$\vec{H}$ = grad area

$2\times \text{area} = bh$

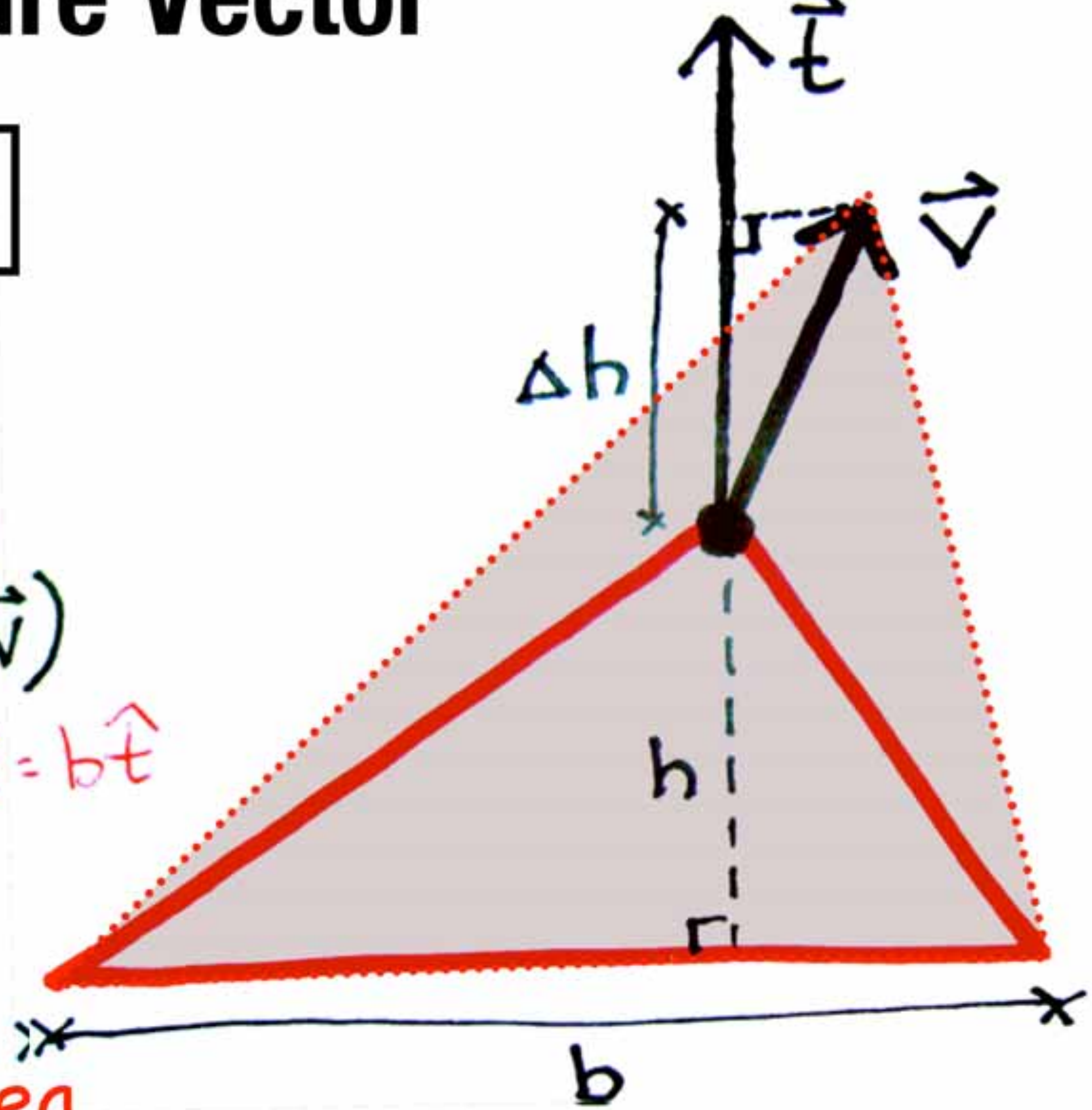
$\Delta\text{area} = b\,\Delta h$

$= b(\hat{t}\cdot\vec{v})$

let $\vec{t} = b\hat{t}$

$= \vec{b}\cdot\vec{v}$

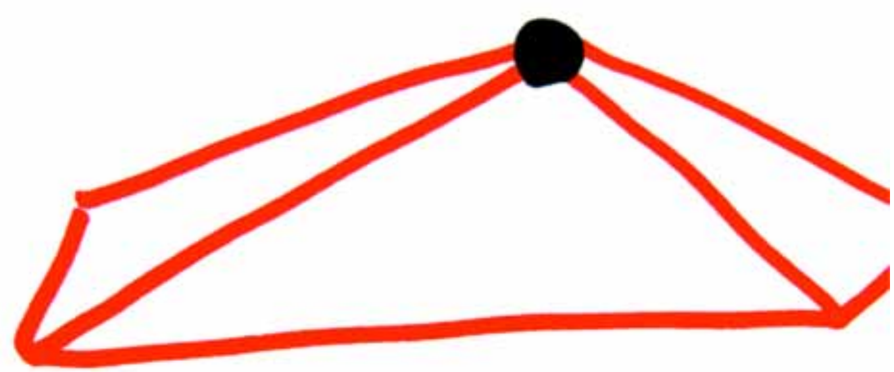
(grad area) v = Δarea



Mean Curvature Vector

Evaluation

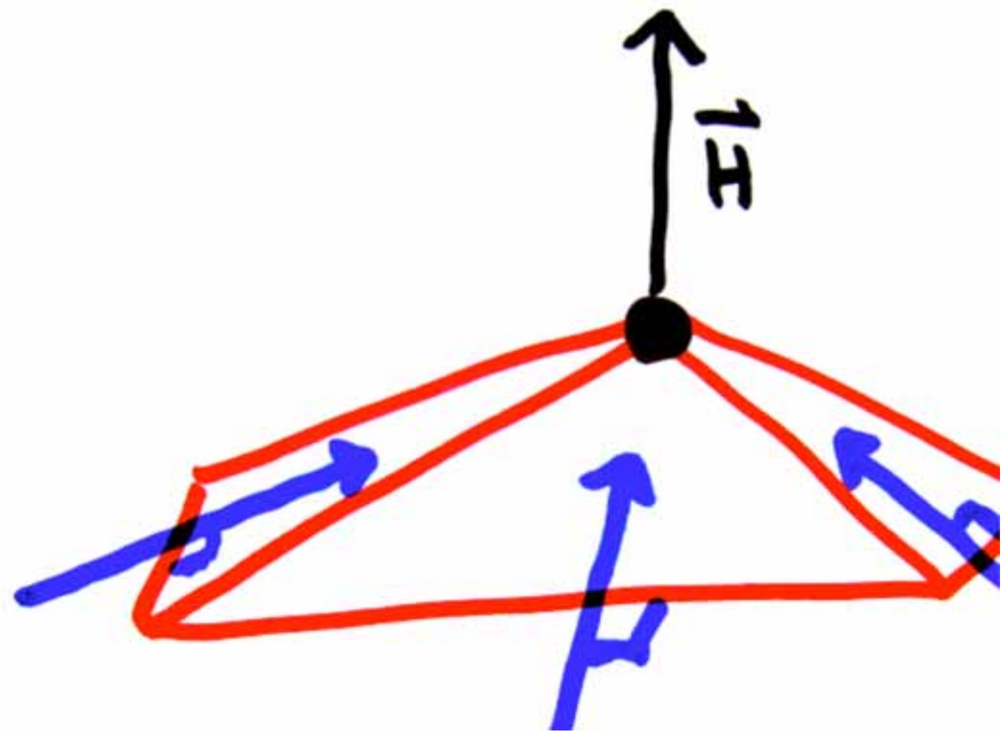
- sum contributions around each vertex



Mean Curvature Vector

Evaluation

- sum contributions around each vertex



Mean Curvature Vector

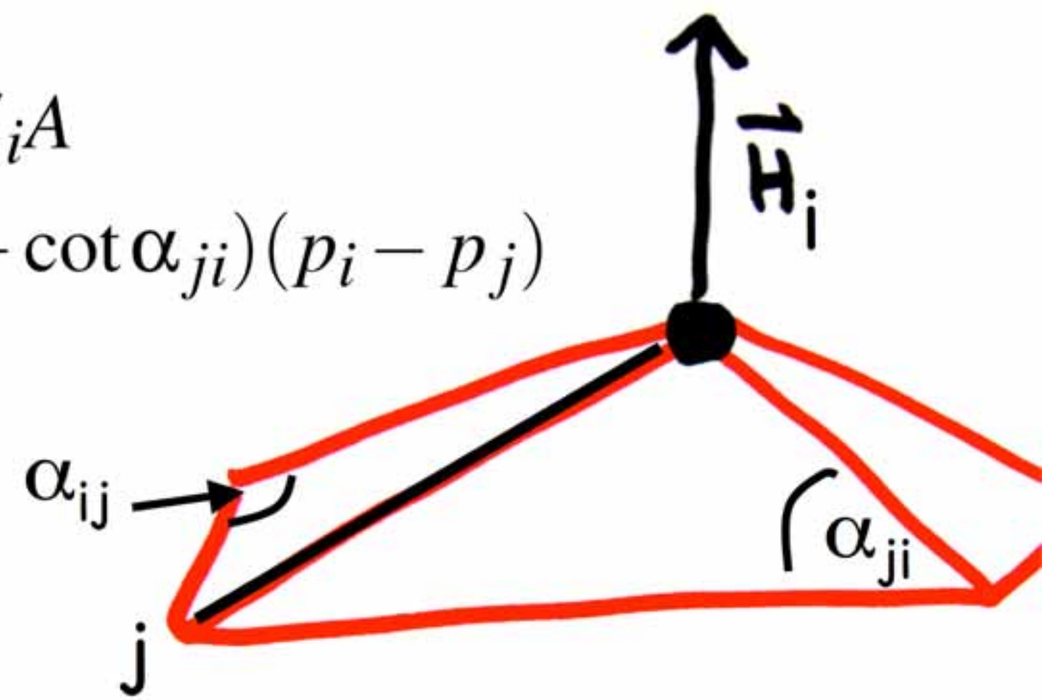
Evaluation

- sum contributions around each vertex

$$2\mathbf{H}_i = \sum_j \mathbf{H}_{e_{ij}} = 2\nabla_i A$$
$$= \sum_j (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j)$$

"cotan formula"

[Pinkall & Polthier]



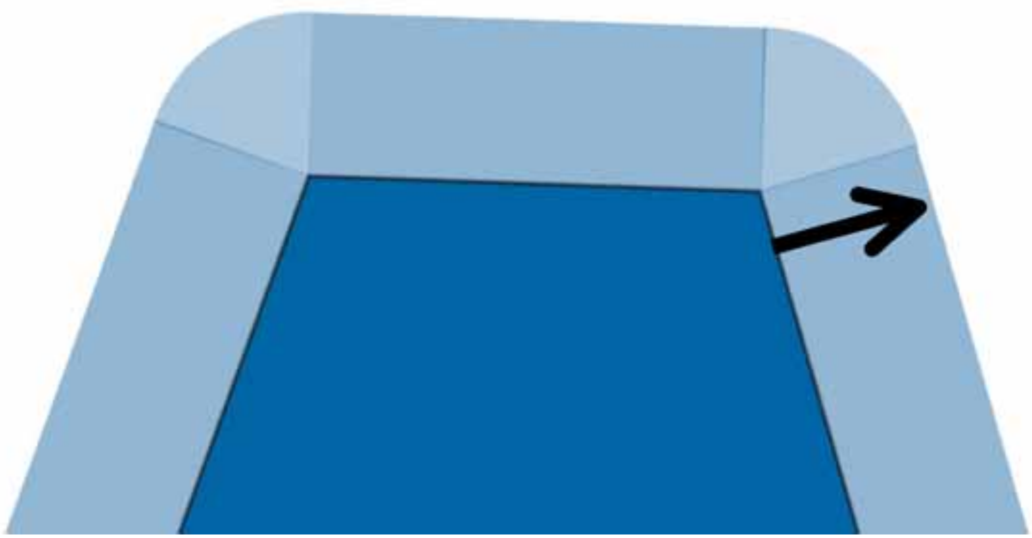
Curvature Measures a la Steiner

Steiner, Cauchy, Hadwiger

- expand a convex set outward by epsilon

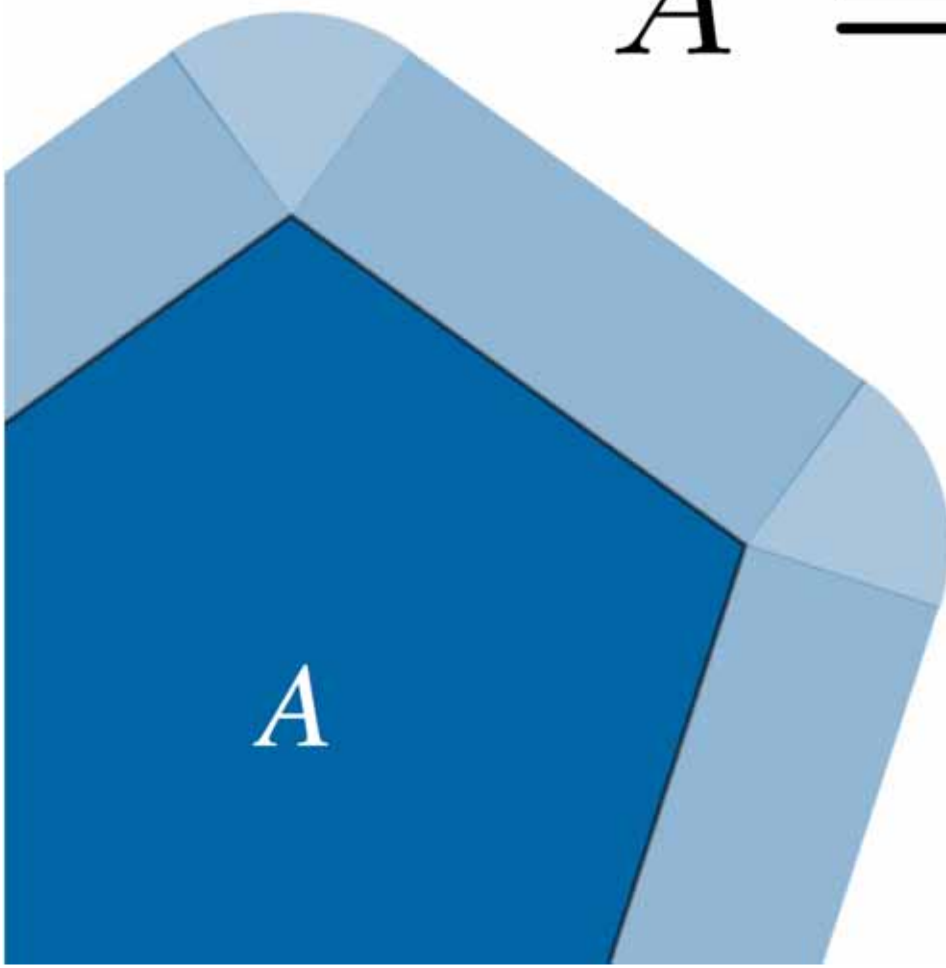
min distance to set

$$\mathcal{K}_\varepsilon = \{x \in \mathbb{R}^n : d(\mathcal{K}, x) \leq \varepsilon\}$$



A Steiner walk-through, 2d

$$A' = A + \dots$$

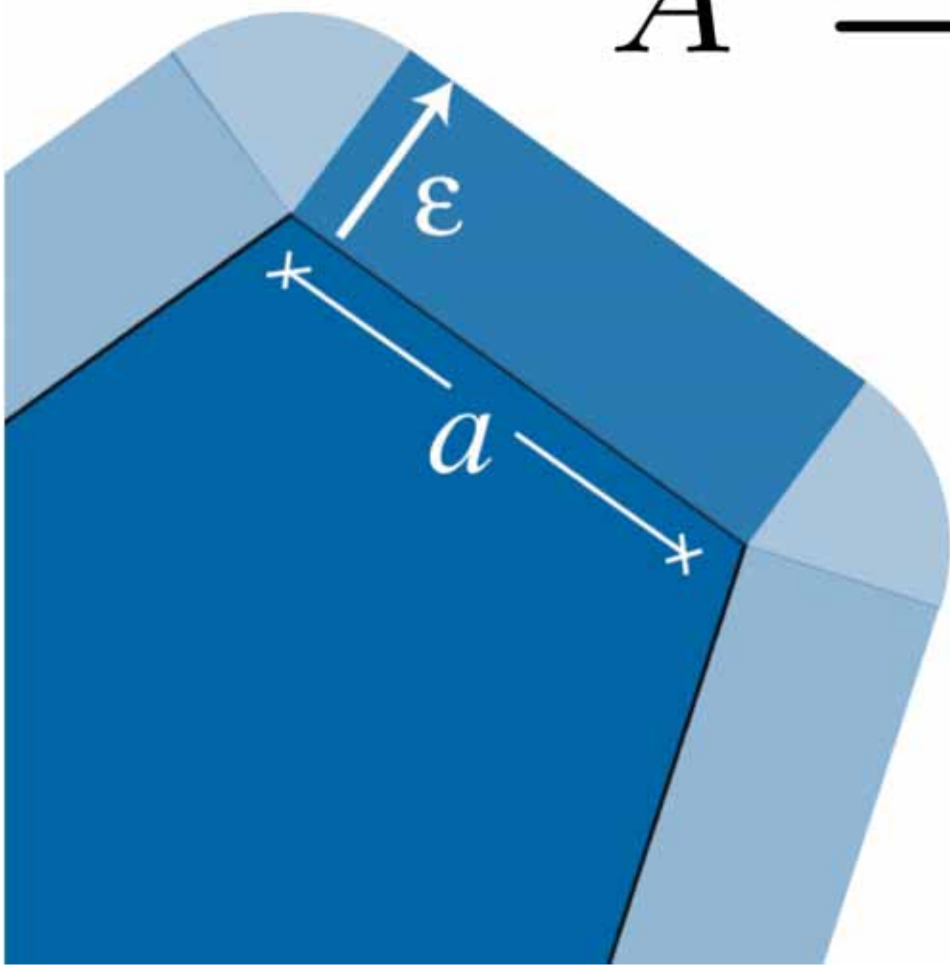


Inflate a planar polygon by epsilon

What is the new area?

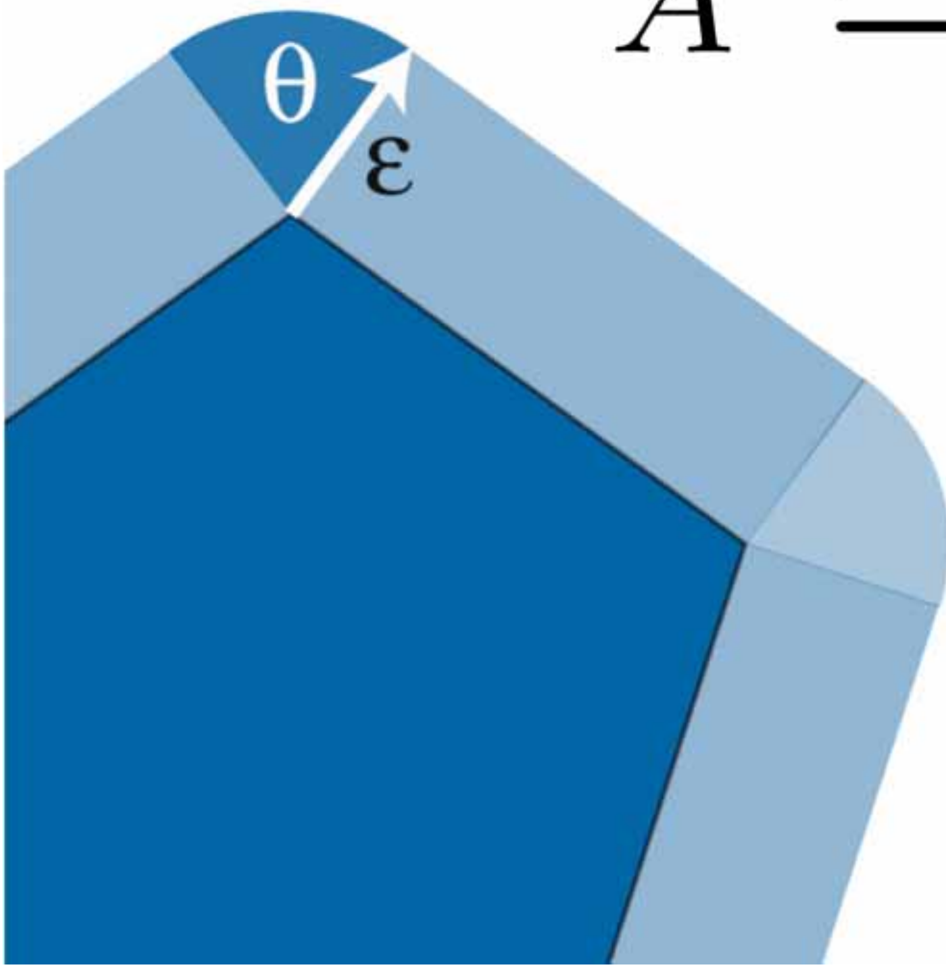
A Steiner walk-through, 2d

$$A' = A + \sum_i \epsilon a_i + \ldots$$



Each edge contributes a rectangle

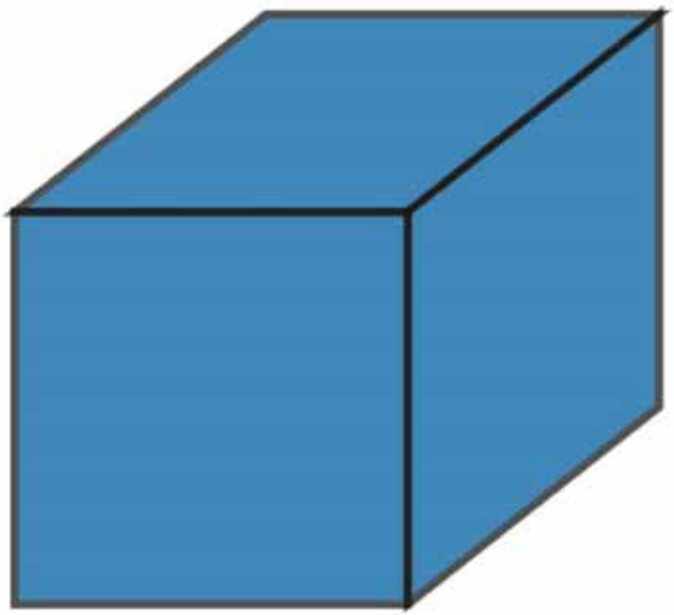
A Steiner walk-through, 2d



$$A' = A + \sum_i \epsilon a_i + \sum_j \epsilon^2 \theta_j$$

Each vertex
contributes a sector

A Steiner walk-through, 3d



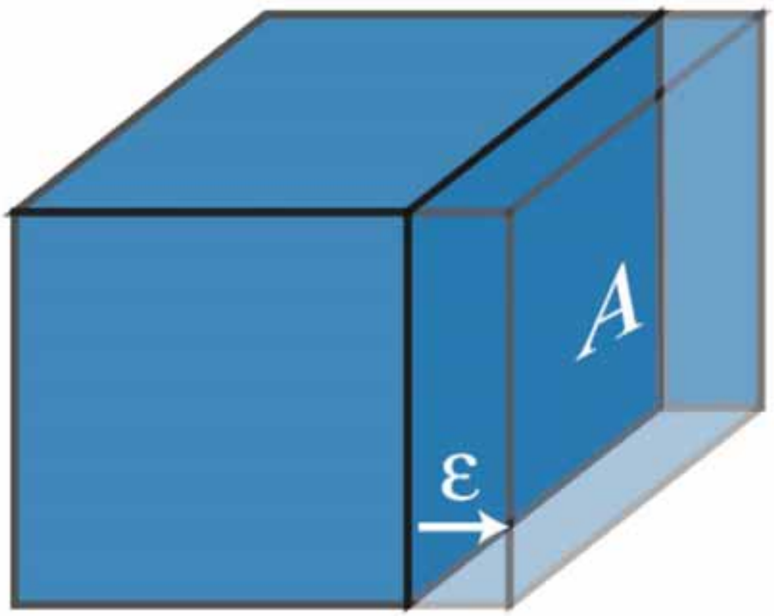
$$V' = V + \ldots$$

Inflate a polyhedron

What is the new volume?

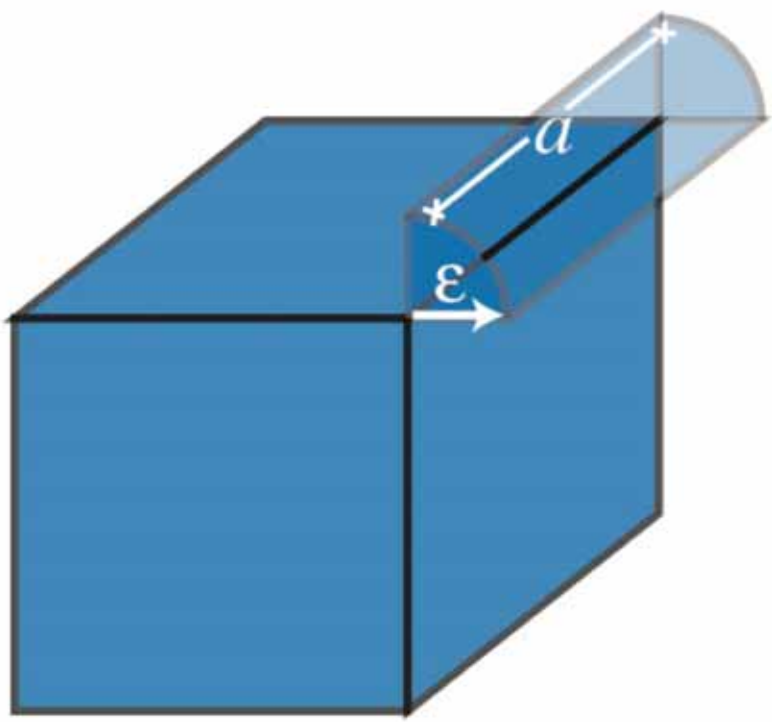
A Steiner walk-through, 3d

$$V' = V$$
$$+\epsilon \sum_i A_i$$



Each face contributes
a parallelotope

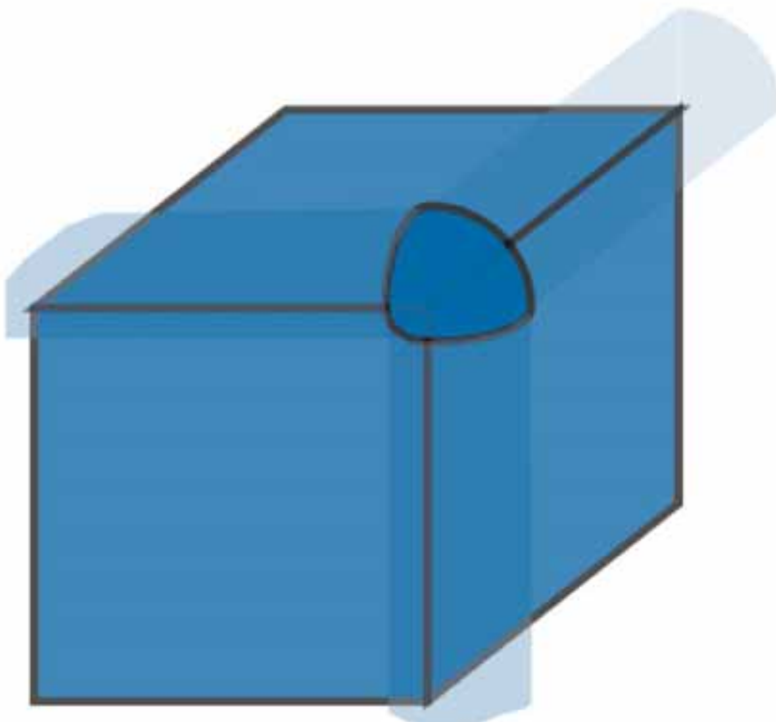
A Steiner walk-through, 3d



$$V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j$$

Each edge contributes a wedge of a cylinder

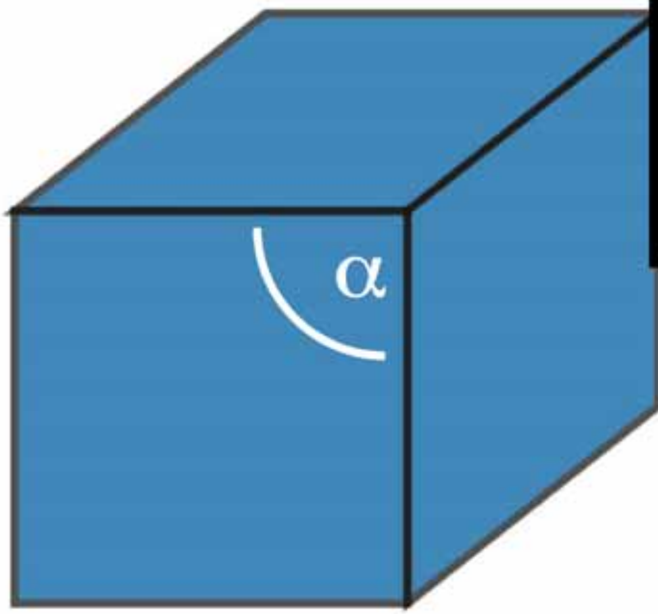
A Steiner walk-through, 3d



$$V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \epsilon^3 \sum_l K_l$$

Each vertex contributes a spherical wedge

A Steiner walk-through, 3d



$$V' = V$$

$$K = 2\pi - \sum_m \alpha_m$$

$$+\epsilon^2 \sum_j a_j \theta_j$$

$$+\epsilon^3 \sum_l K_l$$

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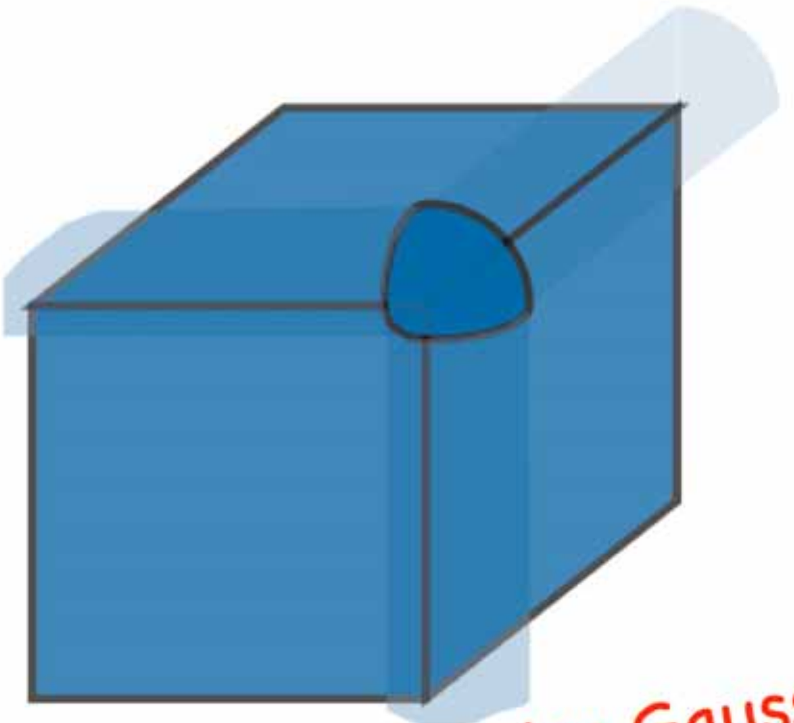

$$V' = V$$

$$+\epsilon \sum_i A_i$$

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$$+\epsilon^3 \sum_l K_l$$

by Gauss-Bonnet =

$$+\epsilon^3 2\pi\chi$$

Each vertex contr al wedge

Inflation in Smooth Setting

Inflate smooth surface,
measure swept area

$$\varepsilon c_1 \int_S H_0 \, \mathrm{d}A + \varepsilon^2 c_2 \int_S H_1 \, \mathrm{d}A + \varepsilon^3 c_3 \int_S H_2 \, \mathrm{d}A$$

A Steiner walk-through, 3d

$$V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \epsilon^3 \sum_l K_l$$

Each vertex contributes a spherical wedge

$H_0 = 1$, $H_1 = (\kappa_1 + \kappa_2)$, $H_2 = \kappa_1 \kappa_2$

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total area

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total area

total mea
curvature

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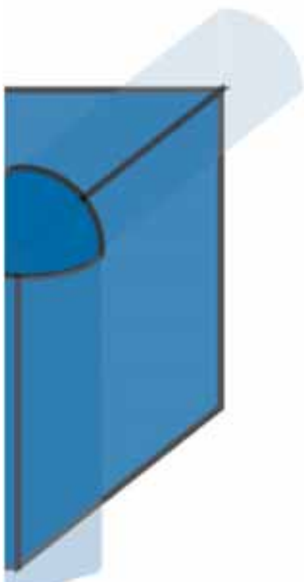
er walk-through, 3d



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ertex contributes a spherical wedge

Setting

$$\varepsilon c_1 \int_S H_0 \mathrm{d}A$$

$$\varepsilon^2 c_2 \int_S H_1 \mathrm{d}A$$

$$\varepsilon^3 c_3 \int_S H_2 \mathrm{d}A$$

total area

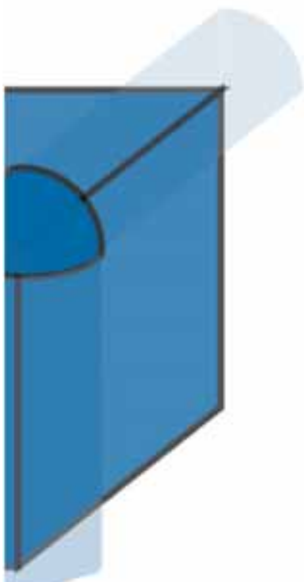
total mea
curvature

total Gau
curvature

$H_0=1$, $H_1=(\kappa_1+\kappa_2)$, $H_2=\kappa_1\kappa_2$

r walk-through, 3d

Setting



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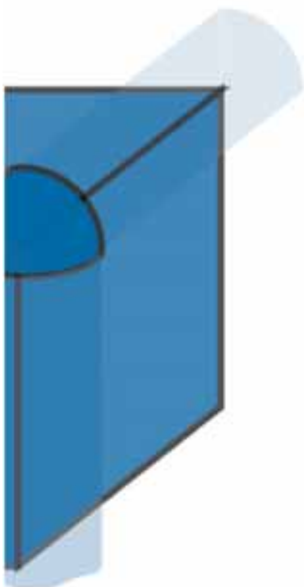

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total area

total mean curvature

total Gau curvature

$H_0=1$, $H_1=(\kappa_1+\kappa_2)$, $H_2=\kappa_1\kappa_2$

r walk-through, 3d

Setting

$$V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \epsilon^3 \sum_l K_l$$

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total area

total mea curvature

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r walk-through, 3d

Setting

$$V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \epsilon^3 \sum_l K_l$$

length × angle

rtex contributes a spherical wedge

$$\varepsilon c_1 \int_S H_0 \mathrm{d}A$$

$$\varepsilon^2 c_2 \int_S H_1 \mathrm{d}A$$

$$\varepsilon^3 c_3 \int_S H_2 \mathrm{d}A$$

total area

total mea[n] curvature

total Gau[ssian] curvature

$H_0=1$, $H_1=(\kappa_1+\kappa_2)$, $H_2=\kappa_1\kappa_2$

r walk-through, 3d

Setting

$$V' = V + \epsilon \sum_i A_i + \epsilon^2 \sum_j a_j \theta_j + \epsilon^3 \sum_l K_l$$

length × angle

scalar not vector

$$\varepsilon c_1 \int_S H_0 \mathrm{d}A \quad \varepsilon^2 c_2 \int_S H_1 \mathrm{d}A \quad \varepsilon^3 c_3 \int_S H_2 \mathrm{d}A$$

total area

total mean curvature

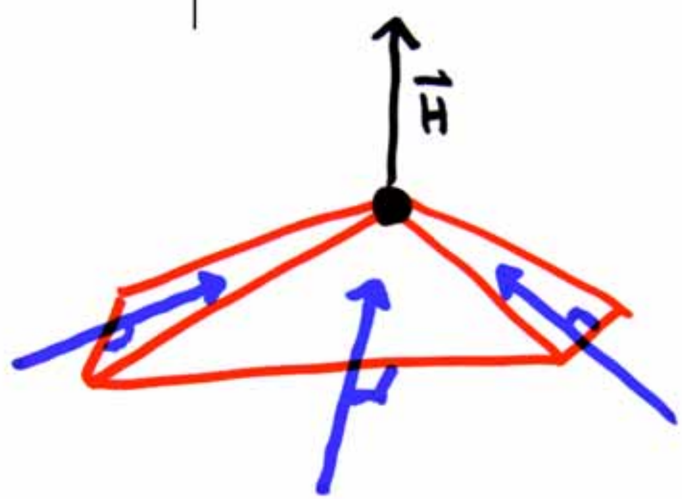
total Gauss curvature

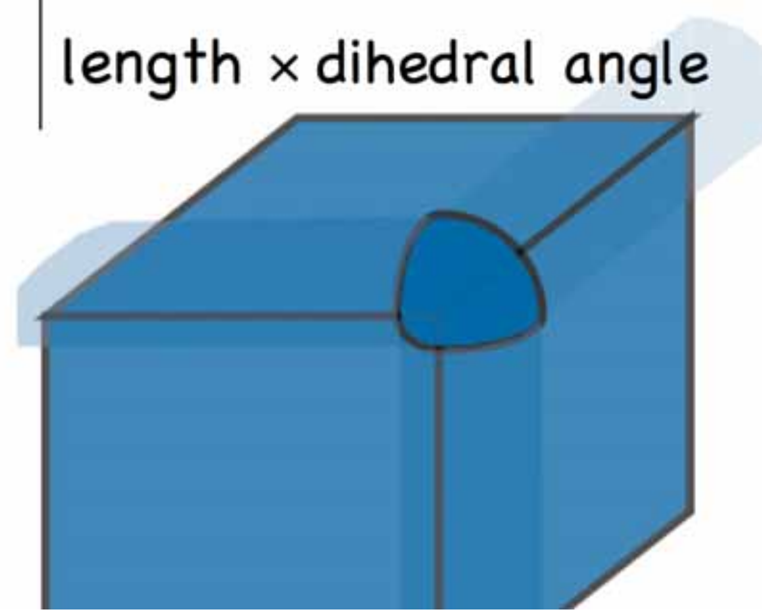
$H_0=1$, $H_1=(\kappa_1+\kappa_2)$, $H_2=\kappa_1\kappa_2$

rtex contributes a spherical wedge

Life & Times of Mean Curvatures

Structure	variational (area)	Steiner polynomial
Species	vector	scalar
Habitat	vertices	edges
Expression	cotan formula	length × dihedral angle





A Plug for Intrinsic Measures

Axiomatic approach

- "what is a reasonable measure?"
- straightforward application to parallelotopes

Geometric probability

- geometry as a dart throwing game

Theorem Hadwiger (1957)

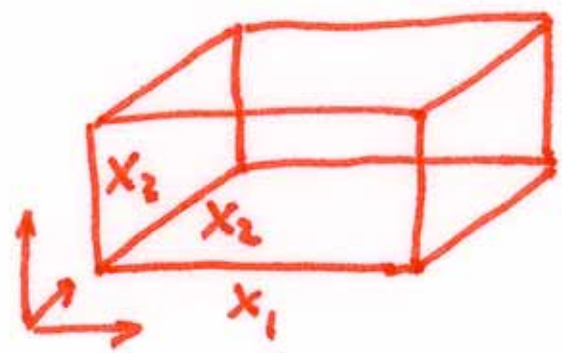
- "These are the only measures you should care about" **Certain restrictions may apply**, Not responsible for exaggerated or untrue claims. If you are elderly, pregnant, or alive, please ask your doctor before using Hadwiger's theorem. Not responsible for incidental, consequential, or any other damages. If you are reading this, you are not paying enough attention to the talk. Stop reading this and listen to me.

What is a reasonable measure?

Properties

- a measure is scalar-valued $\mu(S) \in \mathbb{R}$
- empty set $\mu(\emptyset) = 0$
- additivity $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$
- normalization (parallelotope, P)
  - example: *volume*

$$\mu_n(P) = x_1 x_2 x_3 \ldots x_n$$



Other measures?

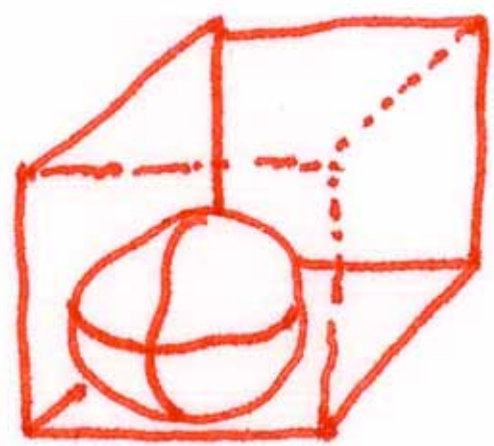
Invariant Measures

Intrinsic volumes

- n measures in n dimensions
- how to generalize to compact convex sets?

Geometric probability

- measure points in set
- probability of hitting set

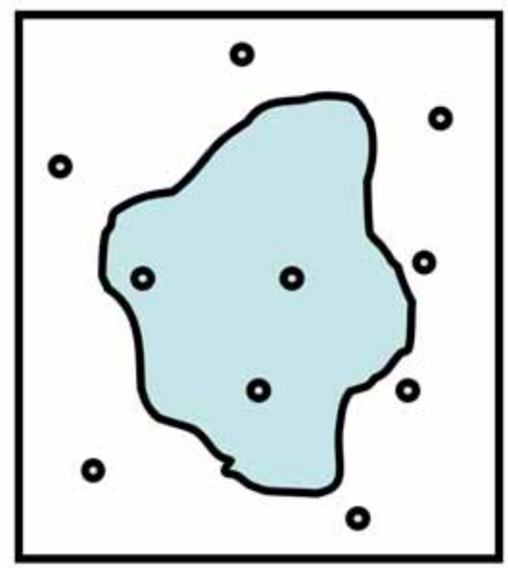


Geometric probability

Blindly throw darts… count number of hits

Darts: k-dim subspaces of n-D

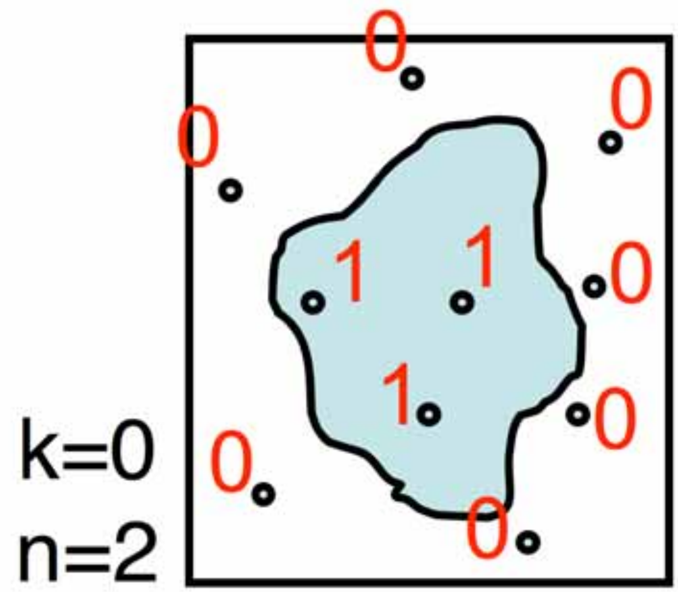
- points
- lines
- planes
- volumes



Geometric probability

Indicator function, $X_C(\omega_i)$

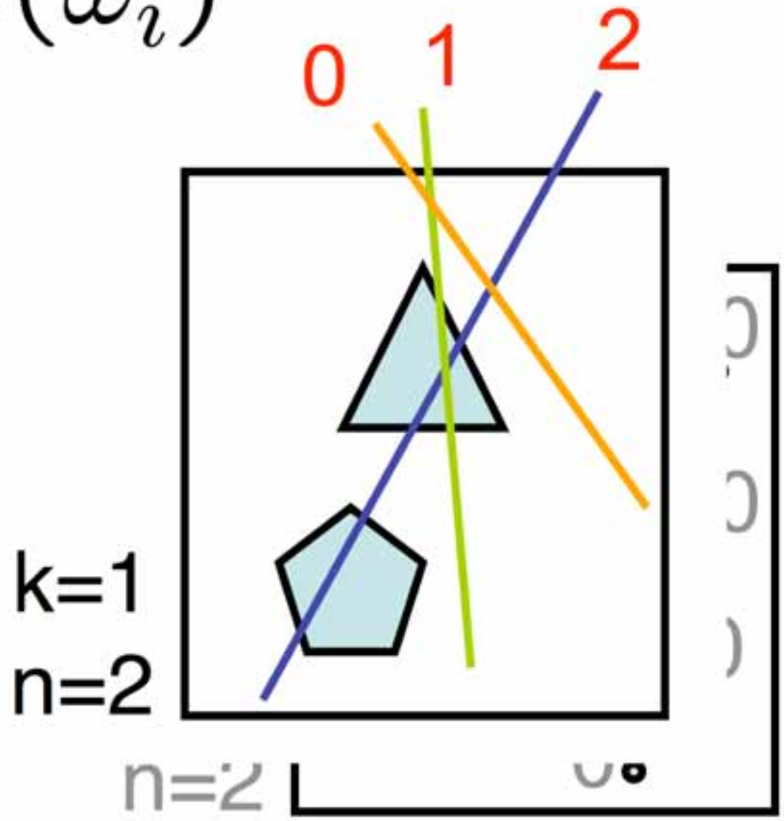
- input: a dart, ω
- output (point dart):
  1 if dart hits body
  0 if dart misses body



Geometric probability

Indicator function, $X_C(\omega_i)$

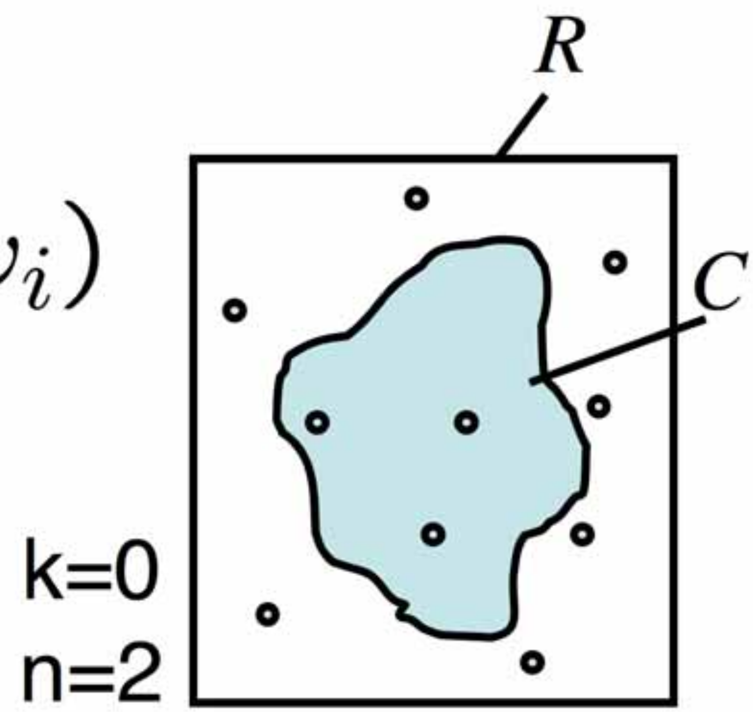
- input: a dart, ω
- output (point dart):
  1 if dart hits body
  0 if dart misses body
- in general,
  output is # hits



Geometric probability

Throw N random darts to estimate area

$$\frac{A_C}{A_R} \simeq \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$



Geometric probability

Throw N random darts to estimate area

Throw all the darts you have…

$$\frac{A_C}{A_R} \simeq \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$

$$A_C \propto \lim_{N->\infty} \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$

Geometric probability

Throw N random darts to estimate area

Throw all the darts you have...

2nd intrinsic volume of C

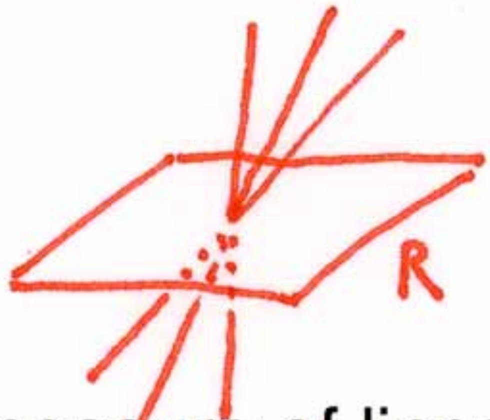
$\mu_2(C)$

measure of darts (k=0,n=2) hitting target C

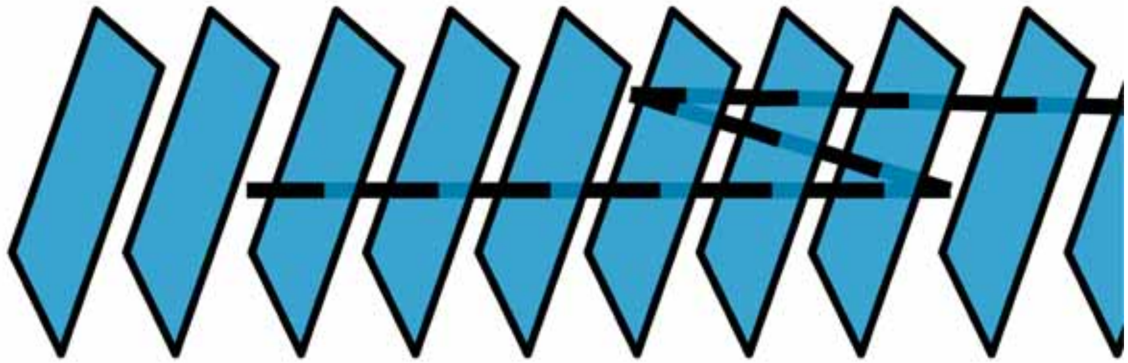
$\lambda_0^2(C)$

$$A_C \propto \lim_{N->\infty} \frac{1}{N} \sum_{i=1}^{N} X_C(\omega_i)$$

Examples of dart-throwing

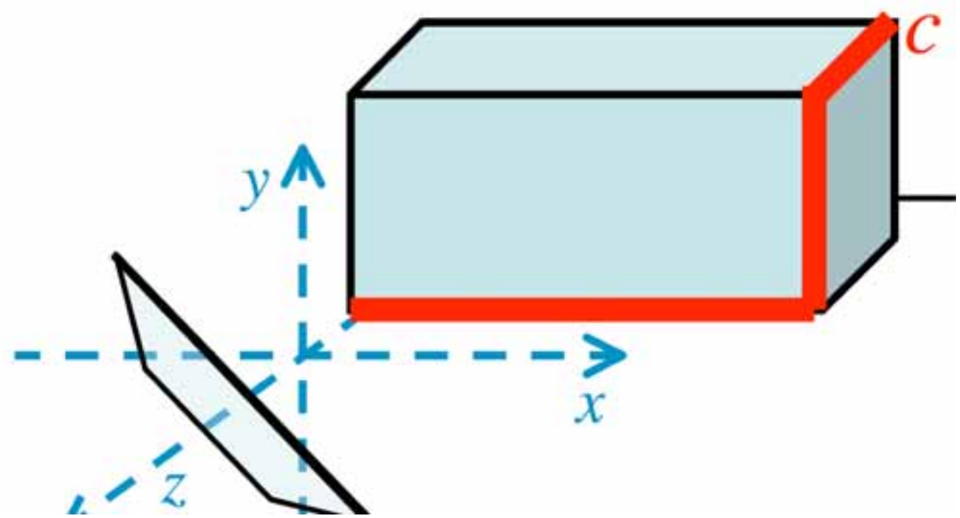


measure of lines
through rectangle
gives surface area



measure of planes
through polyline
gives length

measure of planes
through polytope
gives mean width



Hadwiger (1957)

FUNDAMENAL RESULT

Hadwiger (1957)

These measures
form a basis
for all continuous,
additive,
rigid motion invariant
measures on ring of convex sets.

FUNDAMENAL RESULT

Questions to take home

What can we measure?

- length, angle, area, Gauss & mean curvatures

Where does it live?

- vertex (one-ring), edge (flaps), face

What is its type?

- scalar, vector, tensor...

What structure does it preserve?

- Gauss-Bonnet, area variation, Steiner polynomial

Further Reading

Smooth

Geometry and the Imagination

by Hilbert and Cohn-Vossen

Discrete

DDG Course Notes chapters 1–3

"Introduction to DDG" [Grinspun and Secord]

"What can we measure?" [Schröder]

"Curvature measures for discrete surfaces" [Sullivan]

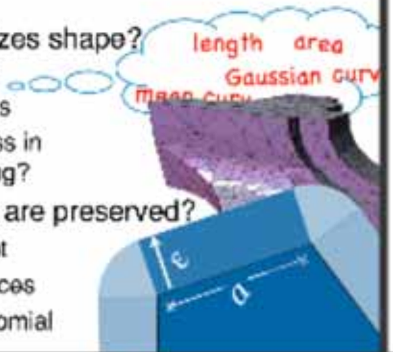
Overview

What characterizes shape?

- brief recall of classic notions
- how to express in discrete setting?

What structures are preserved?

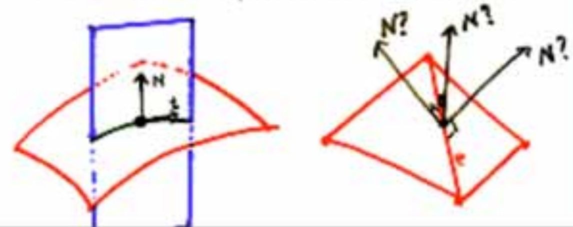
- Gauss-Bonnet
- Minimal surfaces
- Steiner polynomial

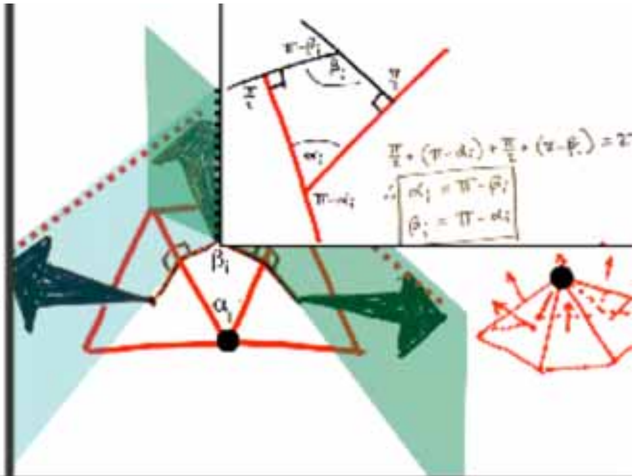


Normal Sections

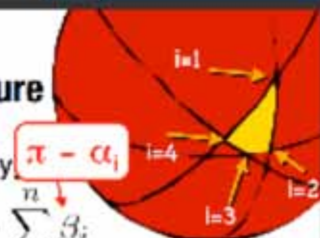
Special family of curves through point P

- choose any plane containing normal
- find the curve of plane/surface intersection





Gaussian Curvature



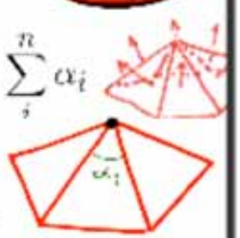
Area of spherical poly

$$A = (2-n)\pi + \sum_i^n \beta_i$$

$$A = (2 - n)\pi + n\pi - \sum_i^n \alpha_i$$

$$A = 2\pi - \sum_i^n \alpha_i$$

total Gauss curvature at vertex

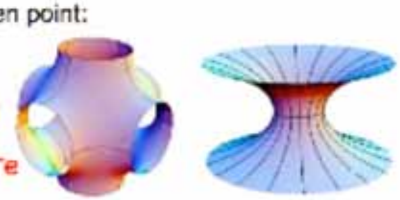


Mean Curvature $(\kappa_1 + \kappa_2)$

Variational structure of mean curvature

- surfaces which minimize area
  - soap bubbles
- at any given point:
  - $k_1 = -k_2$
  - $H = 0$
  - $\mathbf{H} = HN = 0$

mean curvature vector



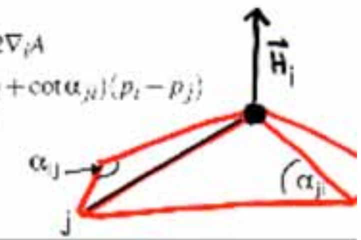
Mean Curvature Vector

Evaluation

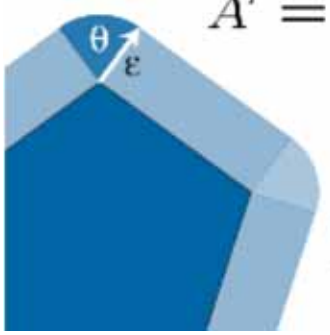
- sum contributions around each vertex

$$2\mathbf{H}_i = \sum_j \mathbf{H}_{e_{ij}} = 2\nabla_i A$$
$$= \sum_j (\cot\alpha_{ij} + \cot\alpha_{ji})(p_i - p_j)$$

"cotan formula"
[Pinkall & Polthier]



A Steiner walk-through, 2d



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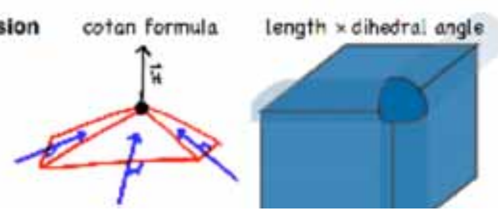

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