DISCRETE DIFFERENTIAL GEOMETRY: An Applied Introduction

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with

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DIFFERENTIAL GEOMETRY

Why do we care?

geometry of surfaces



mothertongue of physical theories
 $E = \int_{S} \alpha + \beta (H - H_0)^2 + \gamma K dA$ computation: simulation/processing



A BIT OF HISTORY

Geometry is the key! studied for centuries





Bobenko and Suris

Cartan, Poincaré, Lie, Hodge, de Rham, Gauss, Noether,...

- mostly differential geometry
 - differential and integral calculus

The study of invariants and symmetries

Getting Started

How to apply DiffGeo ideas? surfaces as collections of samples and topology (connectivity) apply continuous ideas BUT: setting is discrete what is the right way? discrete vs. discretized

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DISCRETIZED

Build smooth manifold structure collection of charts mutually compatible on their overlaps form an atlas realize as smooth functions differentiate away...

DISCRETE GEOMETRY

Basic tool differential geometry metric, curvature, etc. **Discrete realizations** "meshes" computational geom.

graph theory











DISCRETE DIFF.GEOMETRY

Building from the ground up
discrete geometry given
meshes: triangles, tets
more general: cell complex
how to do calculus?

$$\int_{a}^{b} f'(x)dx = f(b) - f(a)$$



DISCR. DIFF. GEOMETRY

Building from the top down high level theorems Riemann mapping $f: \Omega_1 \subset \mathbb{C} \to \Omega_2 \subset \mathbb{C}$ Willmore energy Hamilton's prcple. $\int_{S} (\kappa_1 - \kappa_2)^2 dA$



$$0 = \delta_q \int_0^T L(q(t), \dot{q}(t)) dt$$

WHAT MATTERS?











GAUB MAP: $\vec{n}(s)$



TURNING NUMBER

Number of orbits in Gauß image



TURNING NUMBER THM.

For a closed curve



WARMUP: DISCR. SETTING



INSCRIBED POLYGON:p

Finite number of verticeson curve, orderedstraight edges

LENGTH

Sum of edge lengths



LENGTH



TOTAL SIGNED CURVATURE

Sum of turning angles



DISCRETE GAUB MAP



DISCRETE GAUB MAP



TURNING NUMBER THEOREM

Closed curve

the total signed curvature is an integer multiple of 2π .

proof: sum of exterior angles

$$T_{\kappa} = \sum_{i=1}^{n} \alpha_i = k 2\pi$$

Structure - Preservation

Arbitrary discrete curve

- total signed curvature obeys discrete turning number theorem
- even coarse mesh
- which continuous theorems to preserve?
 - that depends on the application

CONVERGENCE

Consider refinement sequence

- length of inscribed polygon to length of smooth curve
- discrete measure approaches continuous analogue
- which refinement sequence?
 - depends on discrete operator

pathological sequences may exist



L SIGNED CURVATURE

Sum of turning angles



ANOTHER DEFINITION



CURVATURE NORMAL

















MORAL OF THE STORY

Structurepreservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature obeys the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature agree with continuous measures.

THEMES FOR TODAY

What characterizes structure(s)? what is shape? Euclidean invariance what is physics? conservation/balance laws what can we measure? mass, area, curvature, flux, circulation,...

THEMES FOR TODAY

Invariant descriptions integral vorticity 1.6 1.4 1.2 1.0 0.8 0.6 invariance under xforms 20 symmetries give momenta time (seconds) Intrinsic descriptions coord. frame independence

Variational principles solution as optimum

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(b)

(c)

WHAT IT ALL MEANS

Benefits everything is geometric often more straightforward tons of indices verboten! The story is not finished... still many open questions in particular: numerical analysis

THE PROGRAM FOR TODAY

Things we will cover

- surfaces: some basic operators
 - the discrete setting
 - first principles ideas
 - putting them to work







THE PROGRAM FOR TODAY

Things we will cover

a first physics model

W(A, K, H) =



- deformation of a shape
- discrete exterior calculus ∇ $\nabla \times$
 - discrete fluids

discrete shells

structure: vorticity



THE PROGRAM FOR TODAY

Things we will cover conformal geometry conformal parameterizations curvature energies structures in time variational principles discrete time