Discrete Geometric Mechanics and Variational Integrators

Ari Stern, Caltech

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Introduction

- The lesson thus far today:
  - Understanding the geometry of space leads to better discretizations.
  - A geometric viewpoint means understanding symmetries and invariants.
  - What about time discretization?

What is "geometric" mechanics?

- Geometric mechanics looks at the symmetries and invariants of physical systems, such as:
  - conservation of energy
  - conservation of linear/angular momentum
  - variational principles

Benefits of using these geometric methods

- If our simulations/animations of physical systems are faithful to these principles, we can get:
  - greater physical realism
  - at lower computational cost
  - with as-good (or better) accuracy
  - no added complexity in implementation

Ignore geometry at your peril!

- If we ignore these principles?
  - energy can dissipate or “blow up”
  - momentum not conserved
  - need to burn lots of CPU cycles to mitigate these effects and make it look decent ($$$$$$)

Example: the pendulum

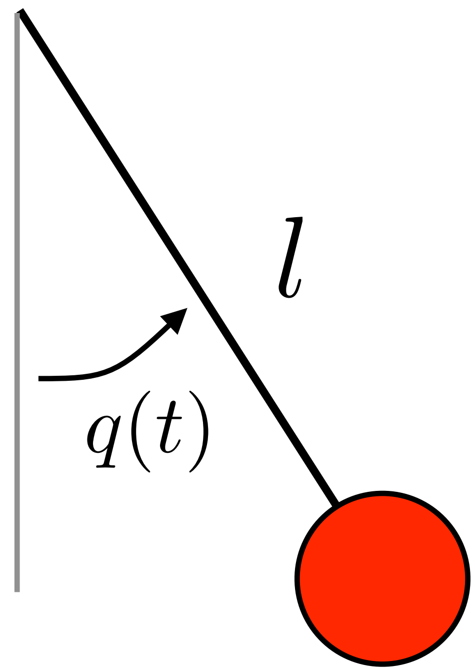
- Denote angle with the vertical at time t by $q(t)$

- Motion described by the differential equation

$$\ddot{q} = -g/l \sin q$$

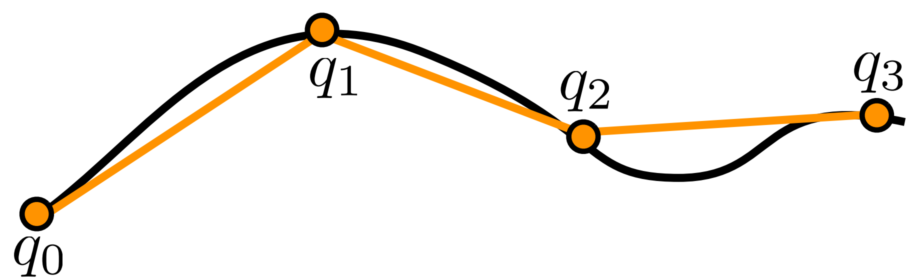
which can be rewritten

$$\begin{aligned} \dot{q} &= v \\ \dot{v} &= -g/l \sin q \end{aligned}$$



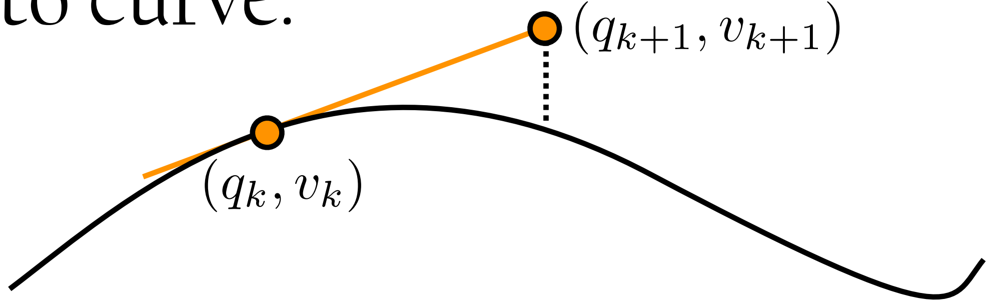
Solving motion numerically

- Nonlinear systems like this are often impractical or impossible to solve exactly
- Use a <u>numerical method</u>:
  - replace the continuous functions $q(t)$ and $v(t)$ by discrete functions q_k and v_k
  - approximate the differential equations, e.g. by first-order Taylor approximation (Euler methods)



Euler methods and Taylor approximation

- Approximate next time step by drawing tangent to curve:



$$q(t_k + h) = q(t_k) + h\dot{q}(t_k) + \mathcal{O}(h^2)$$
$$v(t_k + h) = v(t_k) + h\dot{v}(t_k) + \mathcal{O}(h^2)$$

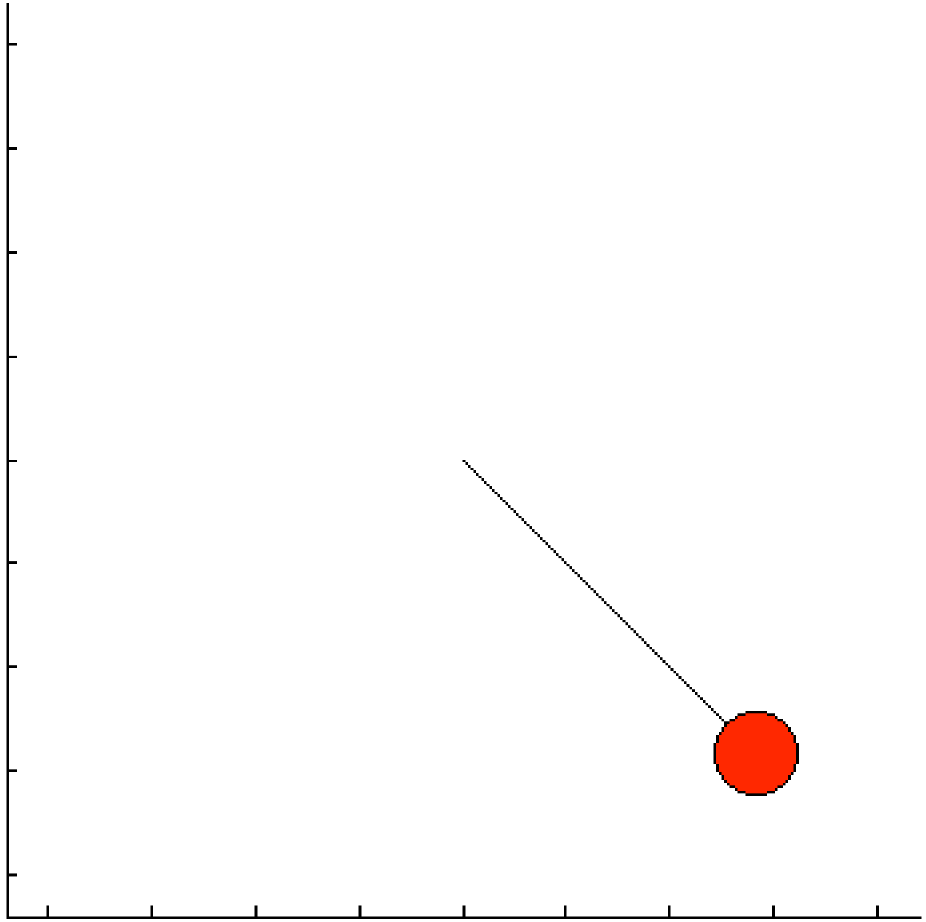
- As $h \to 0$, this approaches the true value.

Explicit Euler method

- Take discrete time steps of equal size $\Delta t = h$

$$\begin{aligned} q_{k+1} &= q_k + h v_k \\ v_{k+1} &= v_k + h(-g/l \sin q_k) \end{aligned}$$

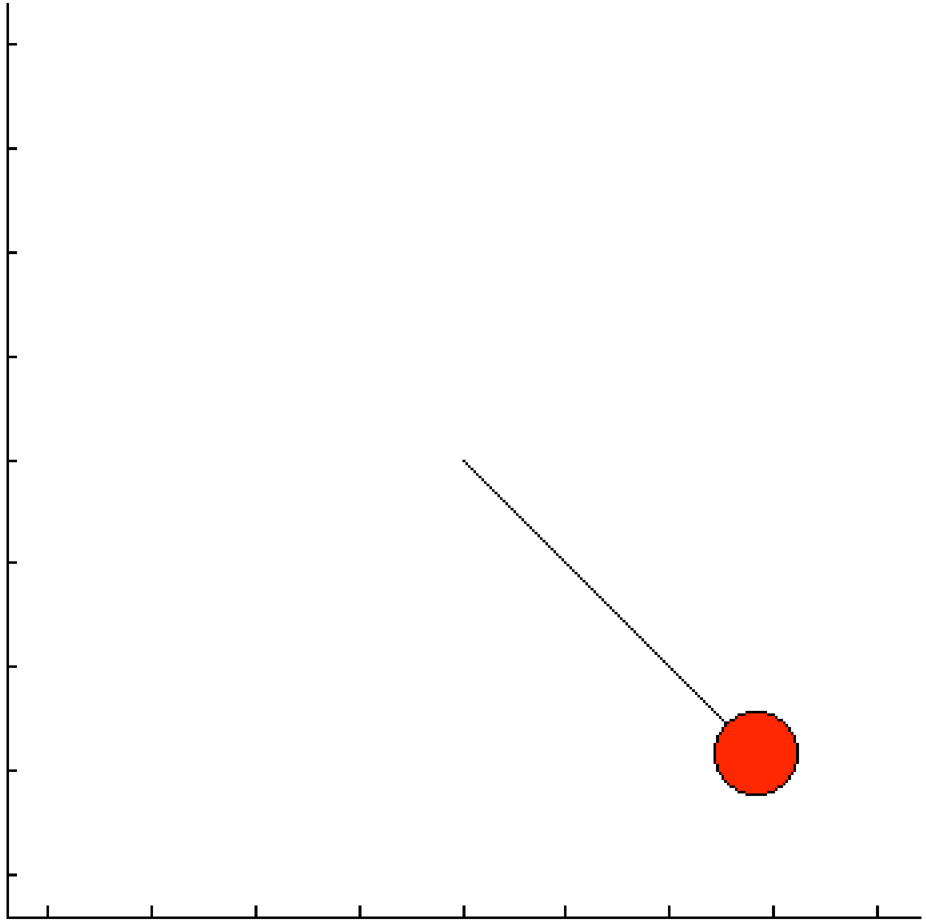
- fast to compute, <u>but</u>:
- energy blowup
- unstable for large time steps



Implicit Euler method

$$
\begin{aligned}
q_{k+1} &= q_k + h v_{k+1} \\
v_{k+1} &= v_k + h(-g/l \sin q_{k+1})
\end{aligned}
$$

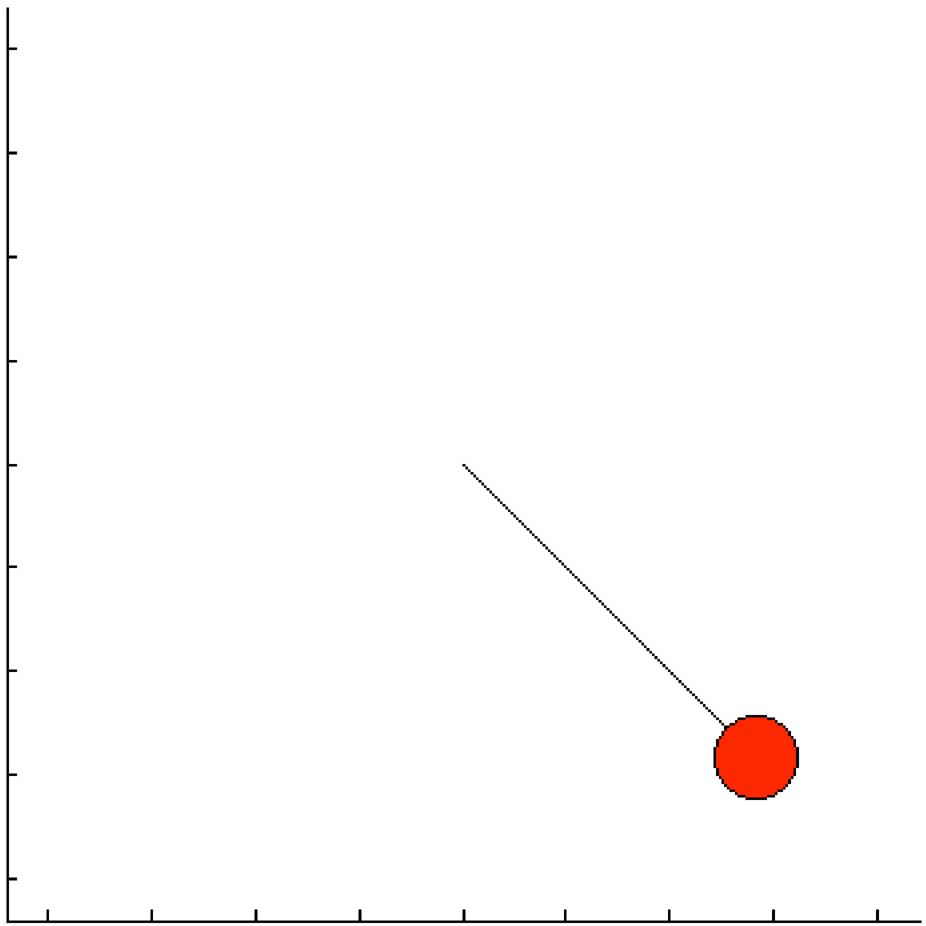
- numerically stable, but:
- energy dissipation
- added computational cost of doing a nonlinear solve at each step



Symplectic Euler method(s)

$$
\begin{aligned}
v_{k+1} &= v_k + h(-g/l \sin q_k) \\
q_{k+1} &= q_k + h v_{k+1}
\end{aligned}
$$

- good energy behavior
- exact same CPU time as explicit Euler
- can still get numerical instability for large time steps (no free lunch!)



Doesn’t matter for some applications

- Why would anyone even use non-symplectic methods?
- In many scientific/engineering problems:
  - only concerned about accuracy at a particular snapshot of time
  - local accuracy instead of global behavior

Different needs for CG

- In computer animation, global behavior and visual/physical plausibility are paramount
- Can often relax local accuracy in favor of better global behavior
- Variational and symplectic integrators let us decouple local accuracy from global behavior
- Can get arbitrarily good accuracy, too; some misconceptions about this

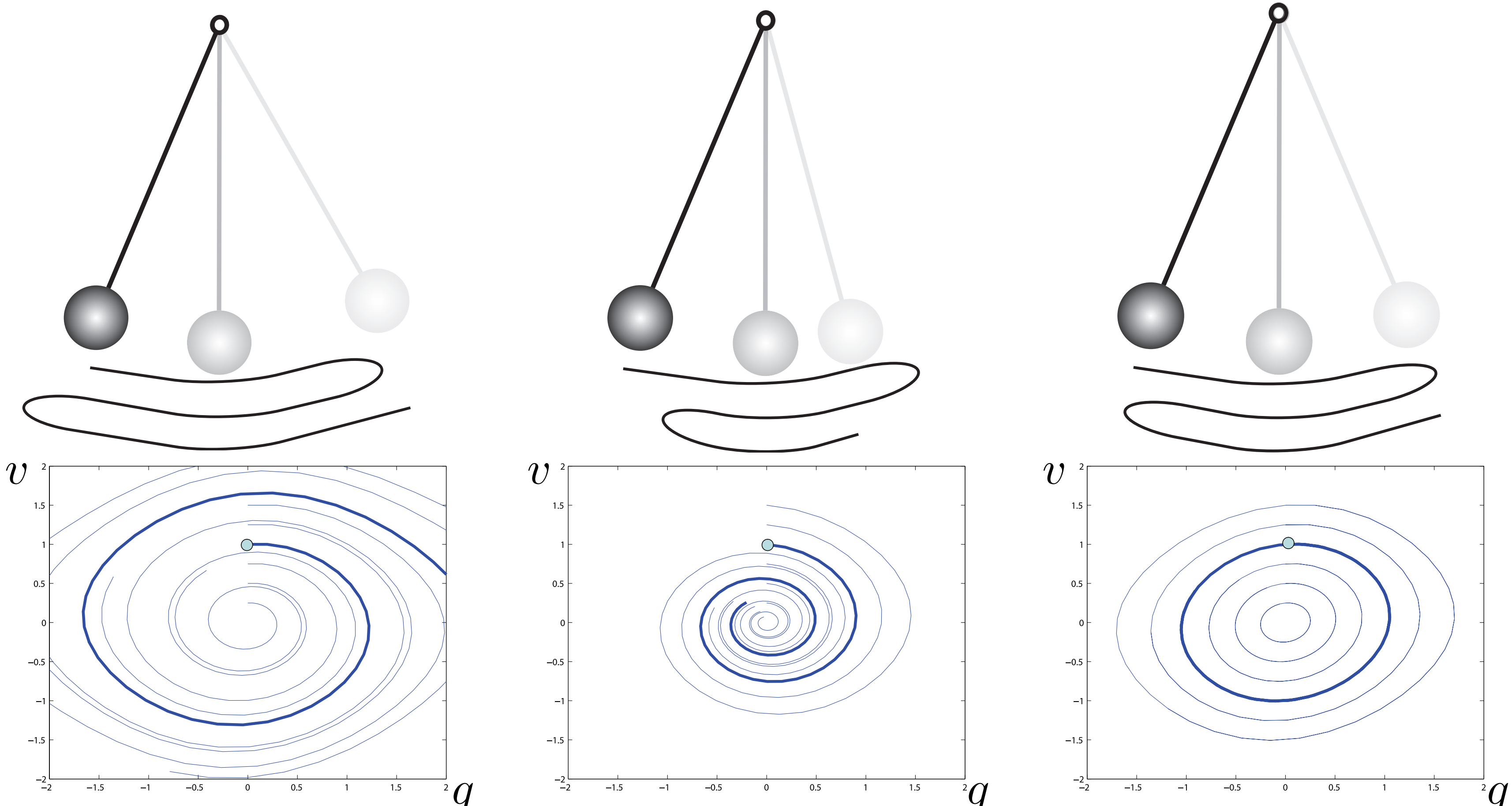
Answering a quick objection

- Don't real systems have dissipation anyway, e.g. friction, damping, air resistance?
- Damping/forcing is <u>step size dependent</u>.
  - Bad for rough-scale "previews."
- <u>Decouple</u> energy behavior from step size.
- Can add damping/forcing to geometric methods in a more precise way.

Line of inquiry

- Why is symplectic Euler so much better?
- How do we come up with other methods like this?
- The approach:
  - Understand why quantities are conserved in continuous systems.
  - Emulate the reasoning for discrete ones.

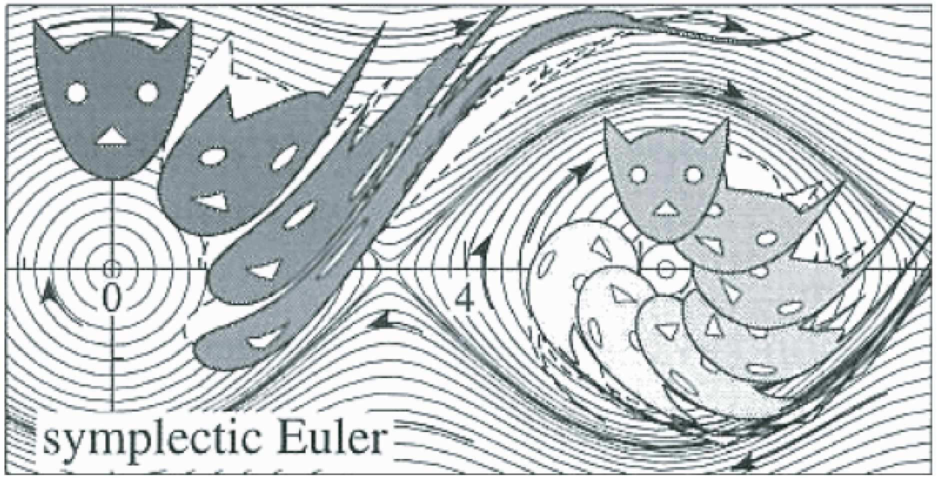
Motion in phase space



explicit implicit symplectic

Symplecticity (oversimplified)

- If we graph trajectories in the phase plane, symplectic methods preserve areas in time.
- This means that a closed loop (e.g. a periodic motion, like the pendulum) won't expand or contract.

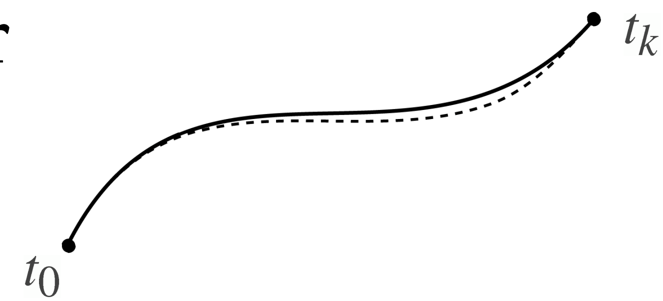


Geometric mechanics

- We need more than just $F = ma$ to explain these invariants (energy, momentum, symplecticity).
- Physical systems follow optimized trajectories (almost like geodesics).
- If our numerical method optimizes a discrete trajectory, then it will have similar geometric properties.

Lagrangian mechanics

- Lagrangian: $L(q, \dot{q}) = \underbrace{K(\dot{q})}_{\text{kinetic}} - \underbrace{V(q)}_{\text{potential}}$
- Action: $S(q) = \int_0^T L(q, \dot{q})\, dt$
- Hamilton's principle: $\delta S(q) = 0$
  - if we "vary" the path slightly, action is unchanged to first order
  - <u>variational principle</u>
  - path extremizes action



Euler-Lagrange equations

- Add a small perturbation ("variation") to the path, which leaves the endpoints fixed

$$q_\epsilon = q + \epsilon \delta q \qquad \delta S(q) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} S(q_\epsilon)$$

- Then the variation of the action is

$$\delta \int_0^T L(q, \dot{q})\, dt = \int_0^T \left(\frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q} \right) dt$$

$$= \int_0^T \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \cdot \delta q\, dt$$

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0}$$

Example: falling object

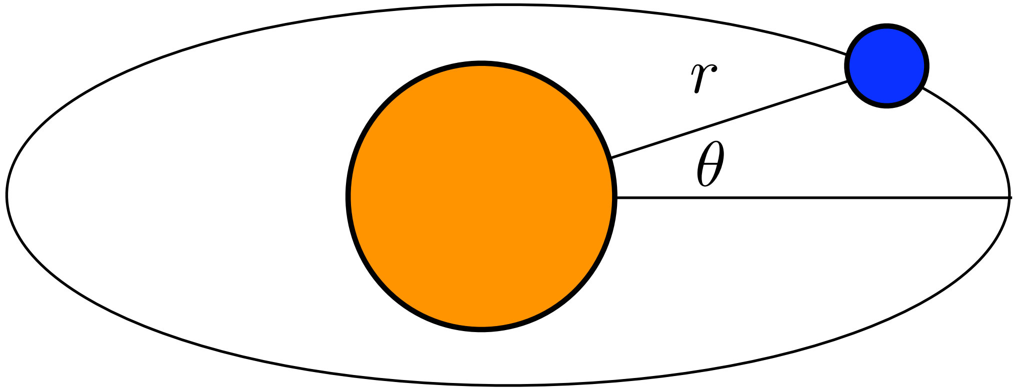
- Kinetic energy: $K(\dot{z}) = \frac{1}{2}m\dot{z}^2$
- Potential energy due to gravity: $V(z) = mgz$
- Lagrangian: $L(z, \dot{z}) = \frac{1}{2}m\dot{z}^2 - mgz$
- Therefore, the equations of motion are

$$-mg - \frac{d}{dt}(m\dot{z}) = 0$$

$$\boxed{\ddot{z} = -g}$$

Properties of the Lagrangian

- Symmetries in the Lagrangian correspond to conserved momenta of the motion (Noether)
    - rotational symmetry = angular momentum

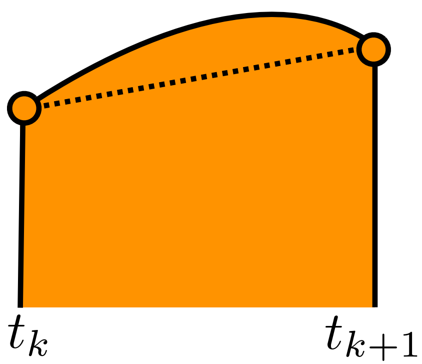


- Lagrangian flows are symplectic

Discrete treatment of Lagrangian mechanics

- Approximate action integral by quadrature rule (midpoint, trapezoid, etc.):

$$L^d(q_k, q_{k+1}) \approx \int_{t_k}^{t_{k+1}} L(q, \dot{q})\, dt$$



  - e.g. midpoint quadrature

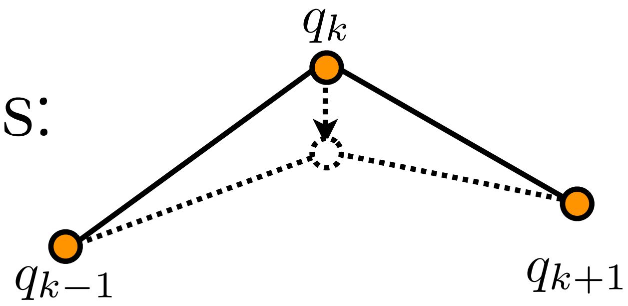
$$= hL\left(\frac{q_k + q_{k+1}}{2}, \frac{q_{k+1} - q_k}{h}\right)$$

- Discrete action <u>sum</u> becomes

$$S^d(q) = \sum_{k=0}^{N-1} L^d(q_k, q_{k+1})$$

Discrete Euler-Lagrange equations

- The discrete action principle is:



$$\delta S^d(q) = 0$$

$$\begin{aligned}\delta \sum_{k=0}^{N-1} L^d(q_k, q_{k+1}) &= \sum_{k=0}^{N-1} \left[D_1 L^d(q_k, q_{k+1}) \cdot \delta q_k + D_2 L^d(q_k, q_{k+1}) \cdot \delta q_{k+1} \right] \\ &= \sum_{k=1}^{N-1} \left[D_1 L^d(q_k, q_{k+1}) + D_2 L^d(q_{k-1}, q_k) \right] \cdot \delta q_k\end{aligned}$$

- Yields the discrete Euler-Lagrange equations

$$\boxed{D_1 L^d(q_k, q_{k+1}) + D_2 L^d(q_{k-1}, q_k) = 0}$$

$$(q_{k-1}, q_k) \mapsto (q_k, q_{k+1})$$

Example: falling object

- Discrete Lagrangian:

$$L^d(z_k, z_{k+1}) = h\left[\frac{1}{2}m\left(\frac{z_{k+1}-z_k}{h}\right)^2 - mg\left(\frac{z_k+z_{k+1}}{2}\right)\right]$$

- Discrete Euler-Lagrange equations:

$$-m\left(\frac{z_{k+1}-z_k}{h}\right) - \frac{1}{2}hmg + m\left(\frac{z_k - z_{k-1}}{h}\right) - \frac{1}{2}hmg = 0$$

$$\boxed{\frac{z_{k+1} - 2z_k + z_{k-1}}{h^2} = -g}$$

Adding forcing/dissipation

- For non-conservative forces, use the discrete Lagrange d'Alembert principle

$$\delta S^d + \sum_{k=0}^{N} \left(F_d^-(q_k, q_{k+1}) \cdot \delta q_k + F_d^+(q_k, q_{k+1}) \cdot \delta q_{k+1} \right) = 0$$

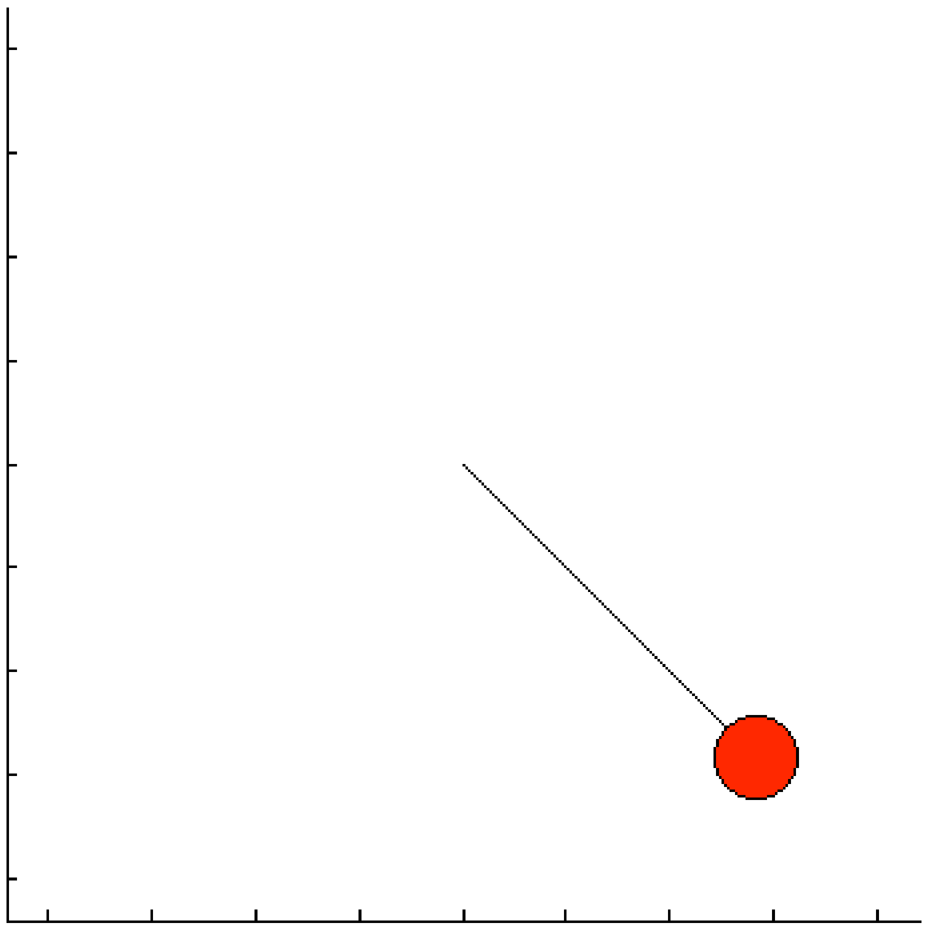
- This gives the forced discrete Euler-Lagrange equations

$$D_2 L^d(q_{k-1}, q_k) + D_1 L^d(q_k, q_{k+1}) + F_d^+(q_{k-1}, q_k) + F_d^-(q_k, q_{k+1}) = 0$$

- Behavior <u>independent of step size</u>.

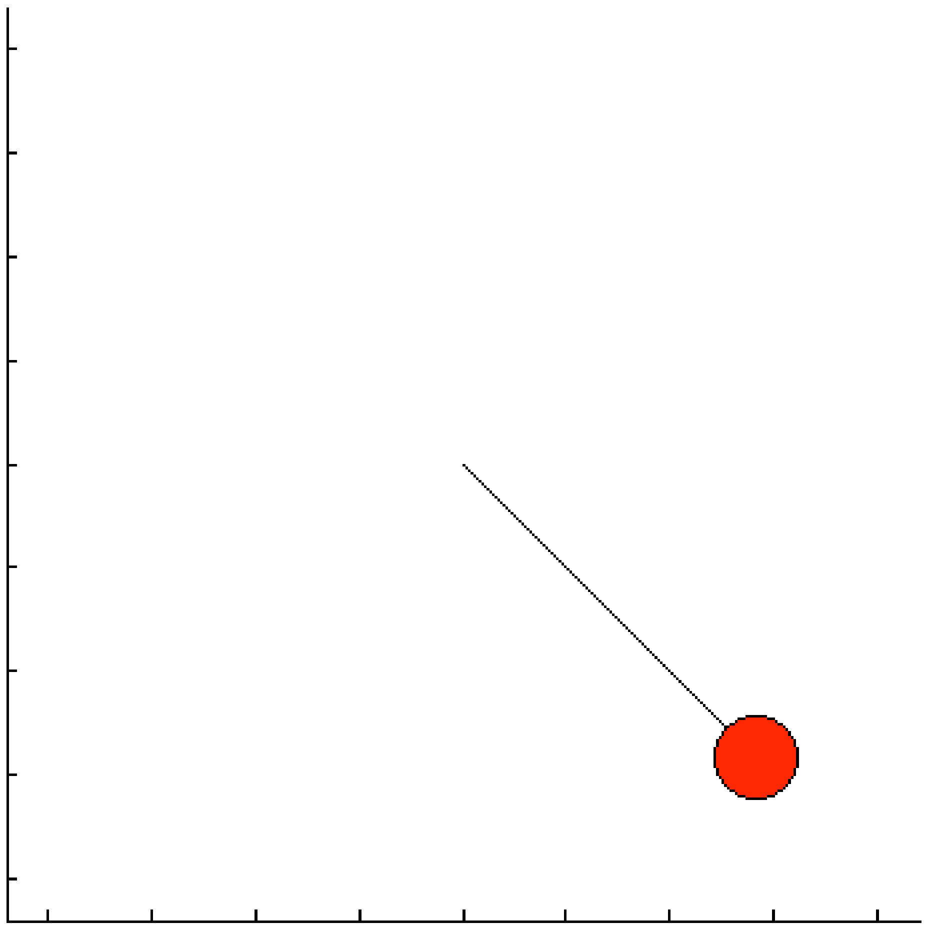
Damped pendulum

- Added damping force proportional to velocity (e.g. air resistance).
- Light damping: coefficient of 0.1



Damped pendulum

- Heavier damping: coefficient of 0.5
- Same number of time steps as previous movie, but <u>different</u> energy behavior.



So why are variational integrators good?

- By respecting the geometric structure of the mechanical system, we automatically get:
  - conservation of momentum,
  - symplecticity,
  - good energy behavior for equal time steps.
  - (Non-uniform and adaptive time stepping are possible, but require more care.)

Examples of good schemes

- Symplectic Euler
- Stormer/Verlet integration
- Midpoint Euler
- Newmark
- Symplectic partitioned Runge-Kutta
- and many others

More recent refinements

- "Lilyan" function (Kharevych et. al.)
  - replace implicit nonlinear solver with more efficient function minimization
- Asynchronous variational integrators (AVI)
  - different time steps at different points in space (where more/less accuracy is needed)

Conclusion

- Variational, symplectic integrators give us:
  - better visual/physical plausibility
  - at lower cost than “traditional” methods
  - respect symmetries and invariants
  - global behavior decoupled from “accuracy”
- Implementing these integrators is often no more difficult than traditional integrators.