Thin Shells
& Curvature-Based Energy

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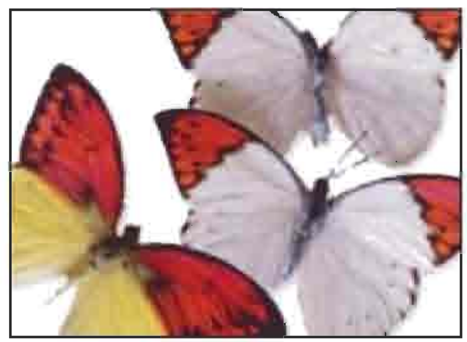
Thin shells and thin plates

Thin, flexible objects

Shells are naturally *curved*

Plates are naturally *flat*


Related work

Researchers in the graphics community:

- Terzopoulos, Bridson, Breen, etc.
  - ad-hoc models for cloth
- Bobenko & Suris, Pai
  - discrete models of elastic curves

[Choi and Ko

Euler's elastica

Early formulation of elastic curves

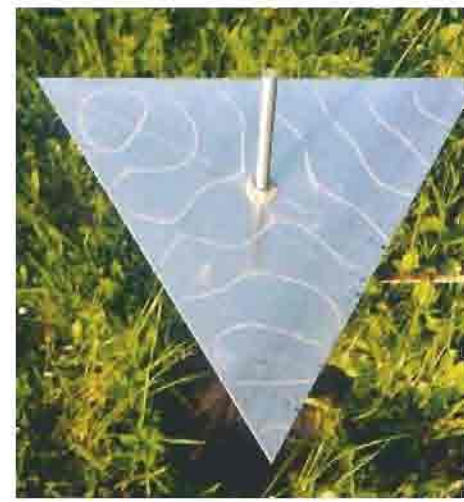
$$E^{\text{bend}} = \int_0^l \kappa(s)^2 ds$$

Bernoulli began generalization to surfaces

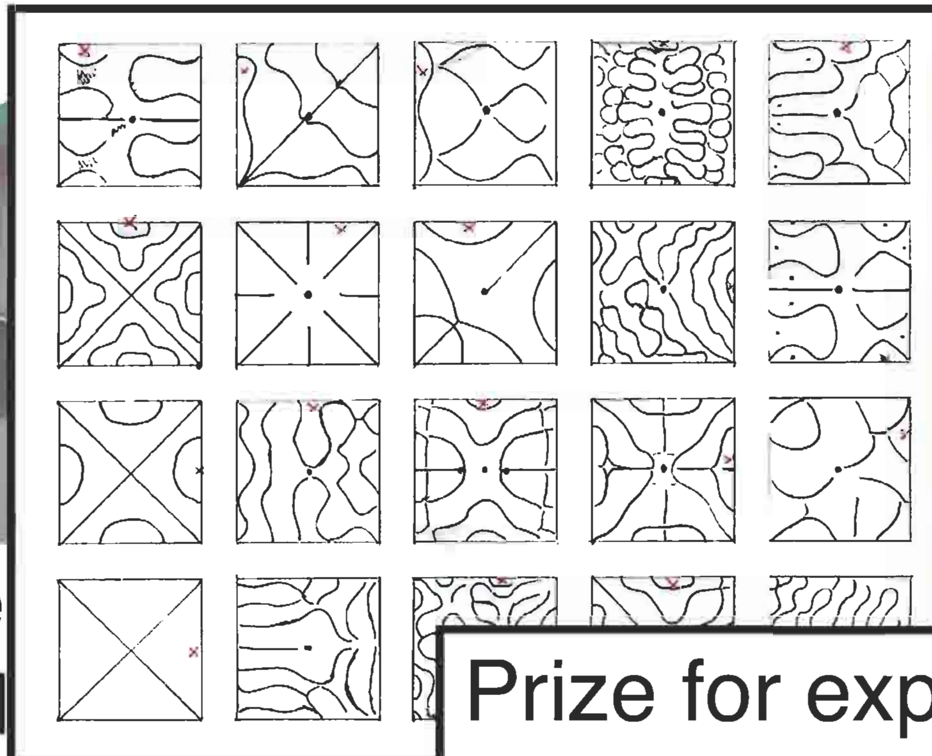
Chladni's vibrating plates

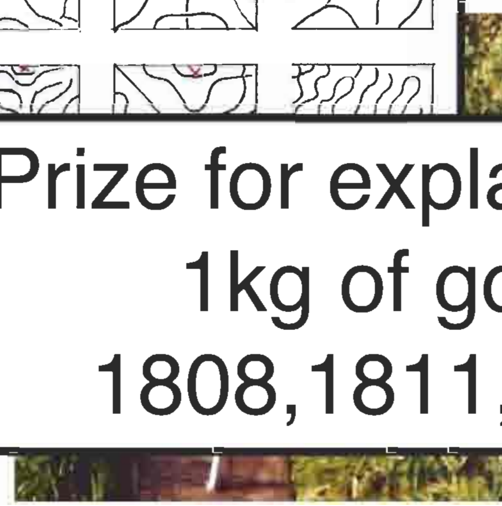
Plate vibrated by violin bow

Sand settles on nodal curves



Chladni's vibrating plates





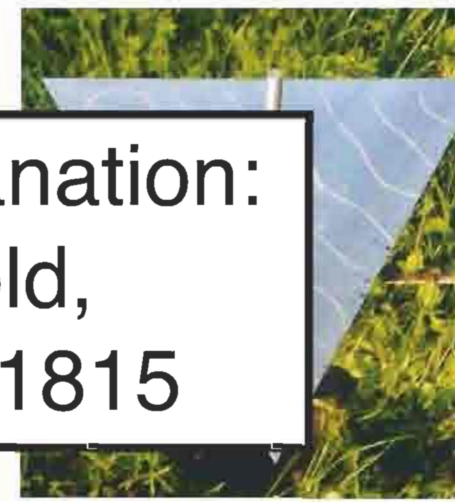


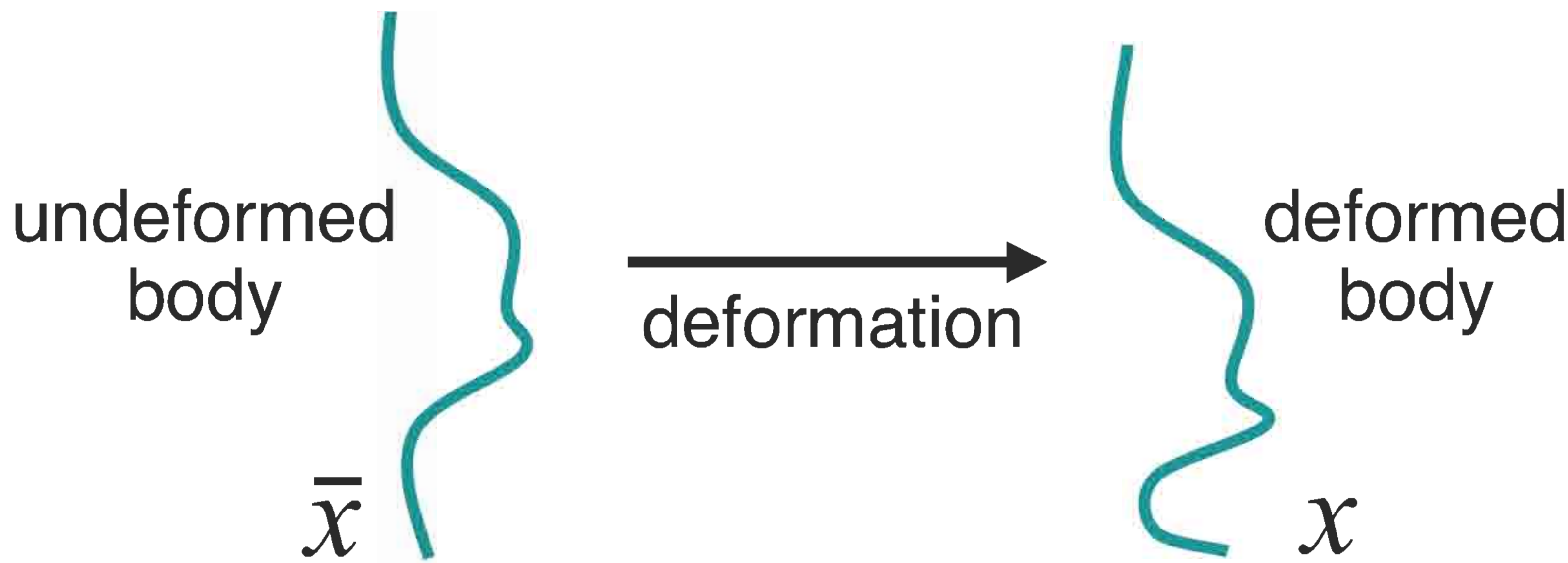
Plate
viol

Sand settles on
nodal curves

Prize for explanation:
1kg of gold,
1808,1811,1815

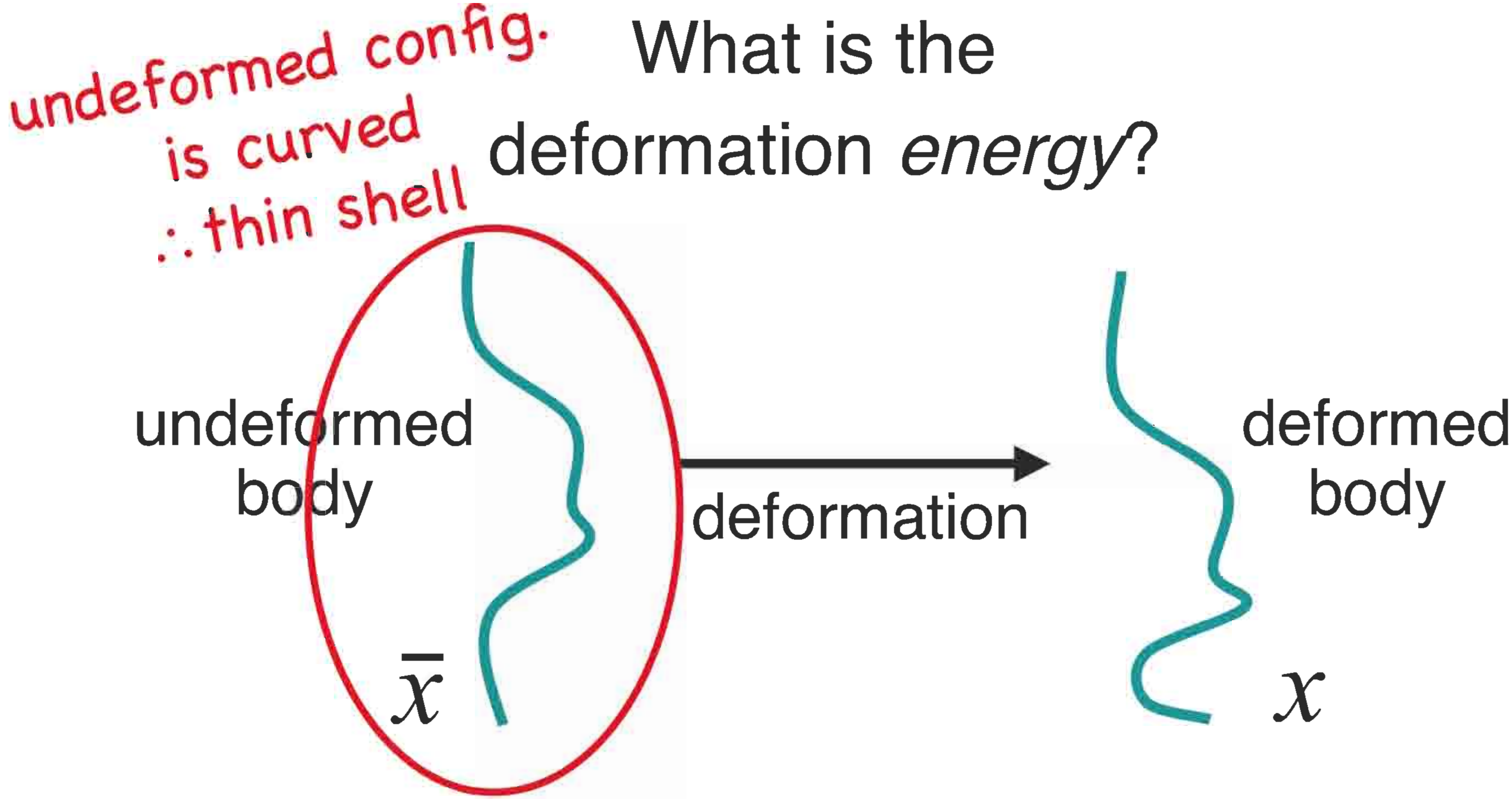
Problem setup

What is the deformation *energy*?



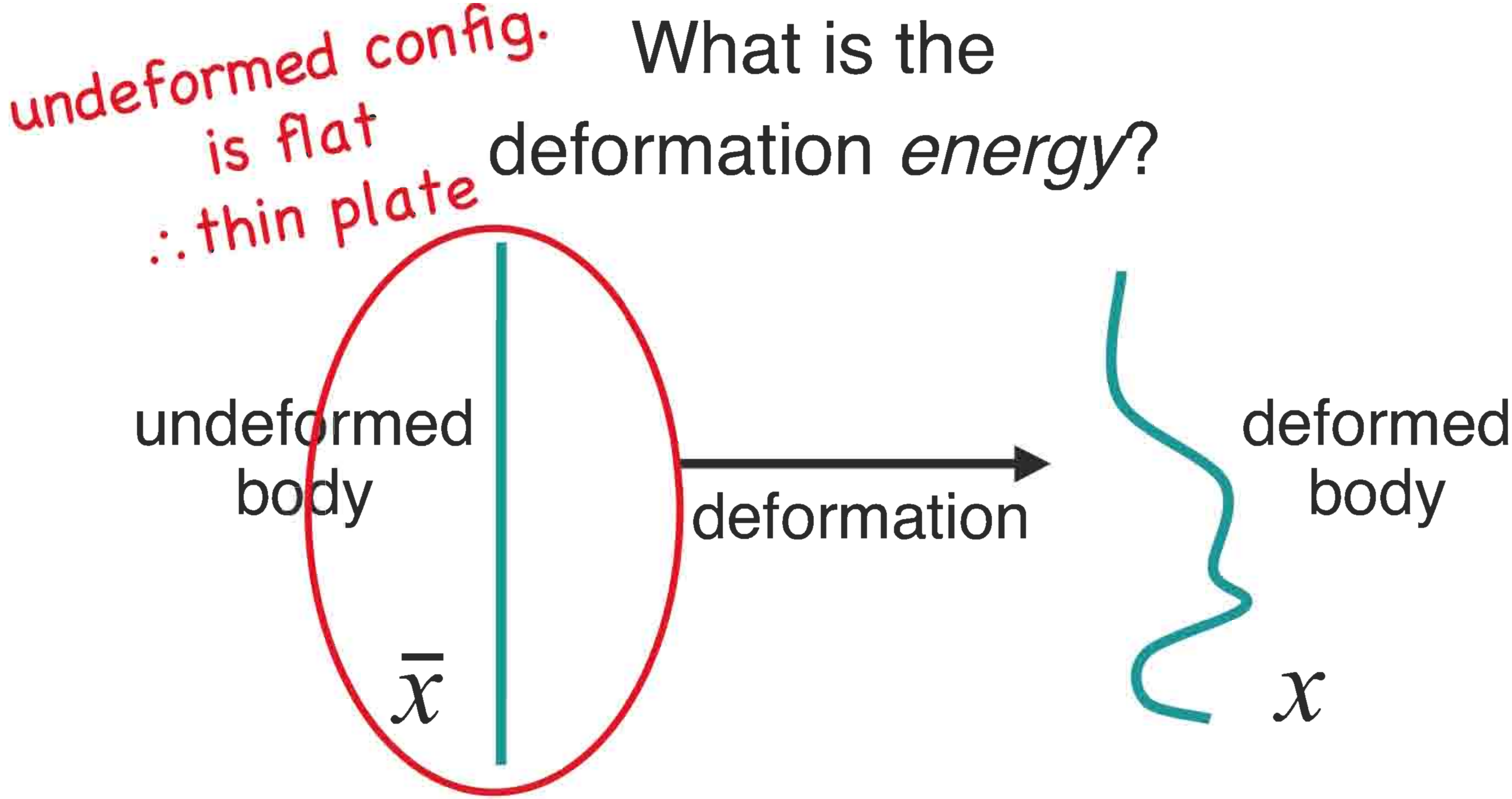
Problem setup

What is the deformation *energy*?



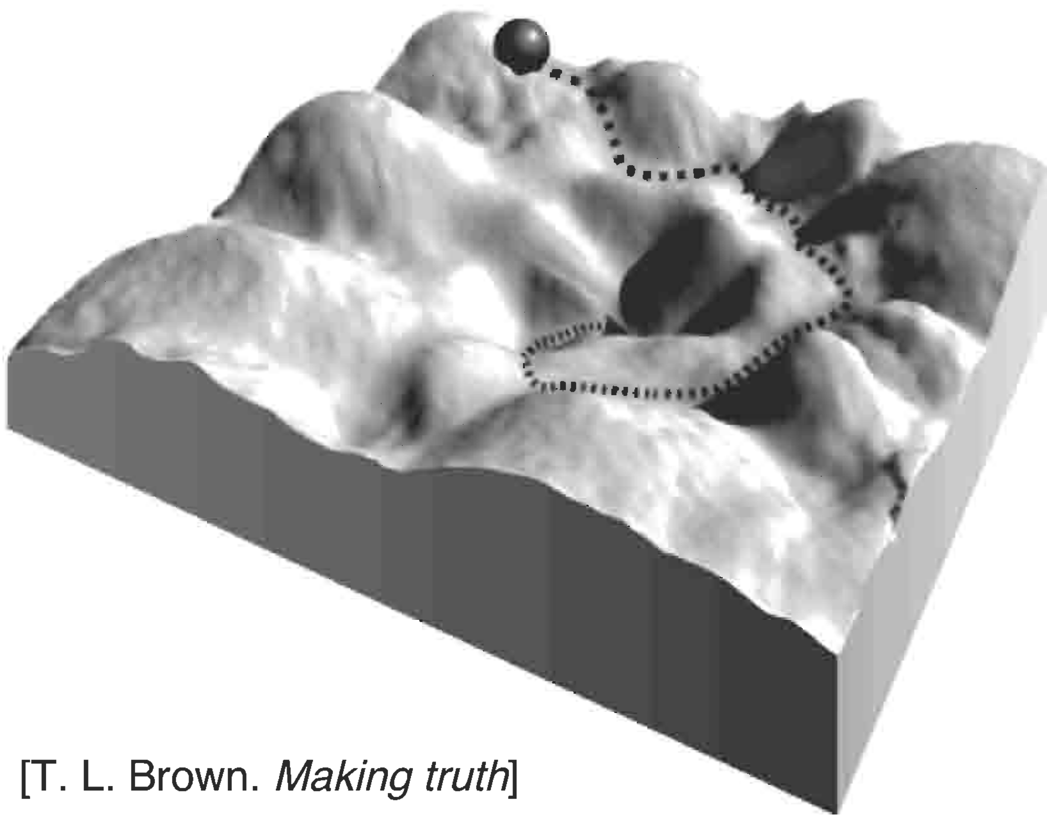
Problem setup

What is the deformation *energy*?



Problem setup

Energy is a non-negative scalar function

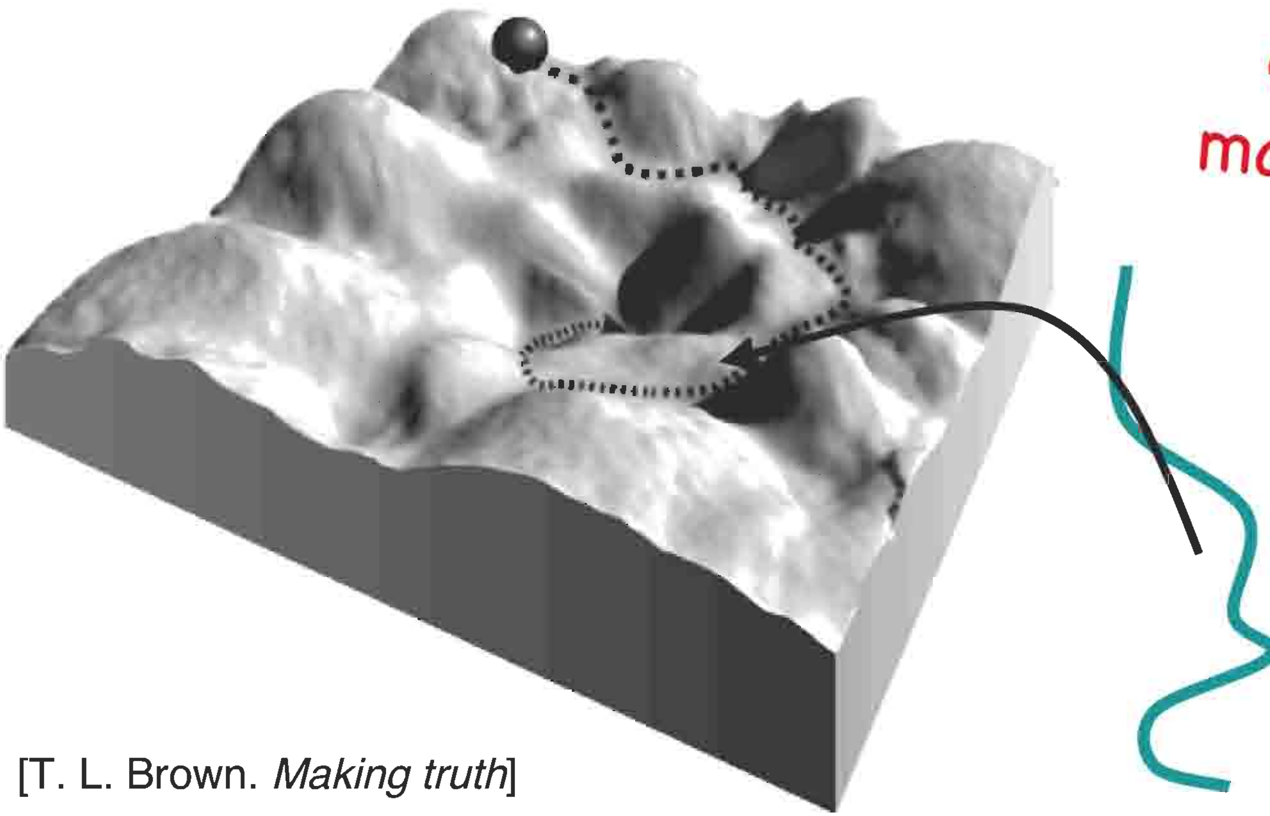


[T. L. Brown. *Making truth*]

Problem setup

Energy is a non-negative scalar function

real number, coordinate-frame invarian



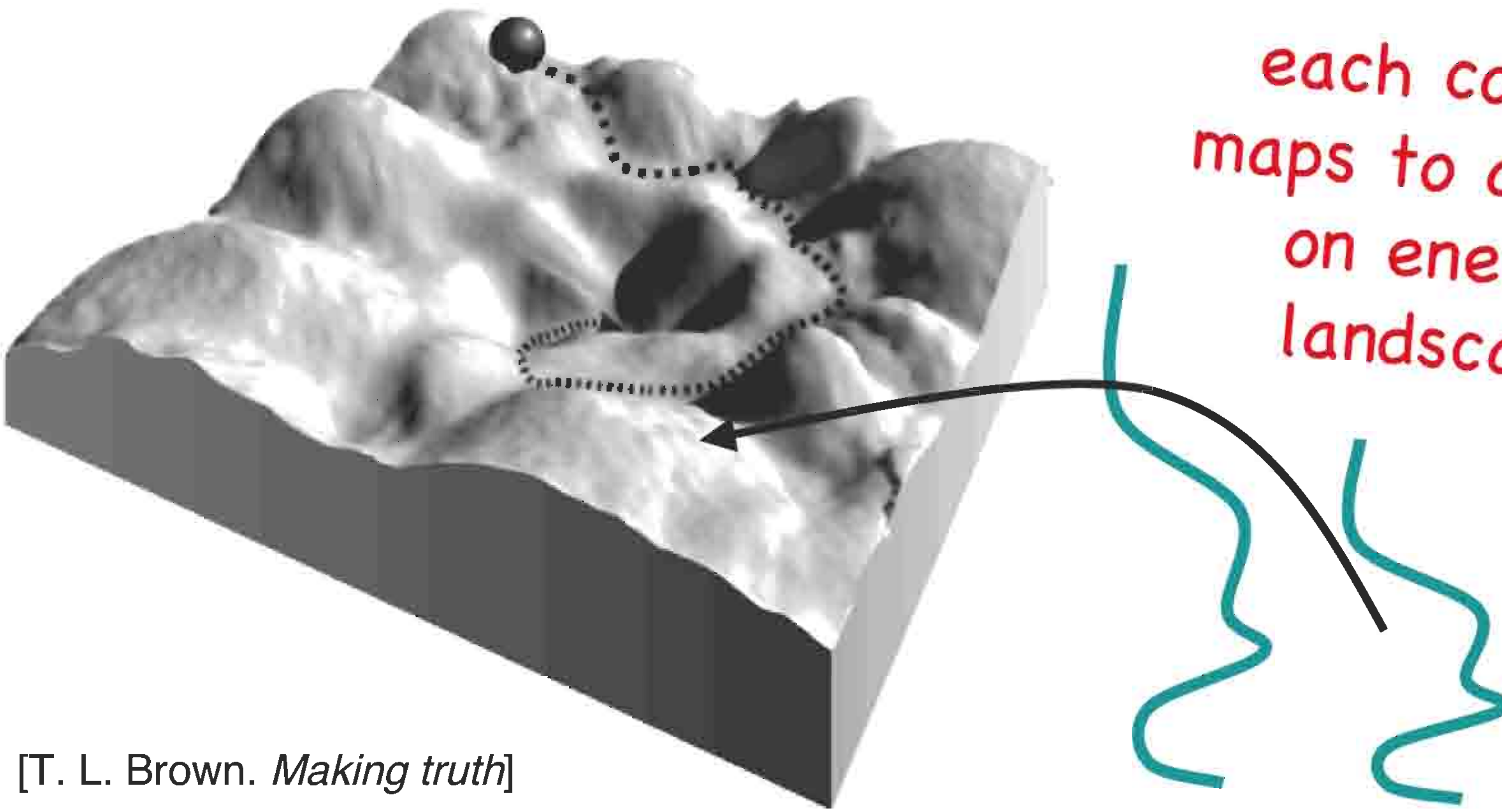
each config- maps to a poi on energy landscape

[T. L. Brown. *Making truth*]

Problem setup

real number,
coordinate-frame invarian

Energy is a non-negative scalar function

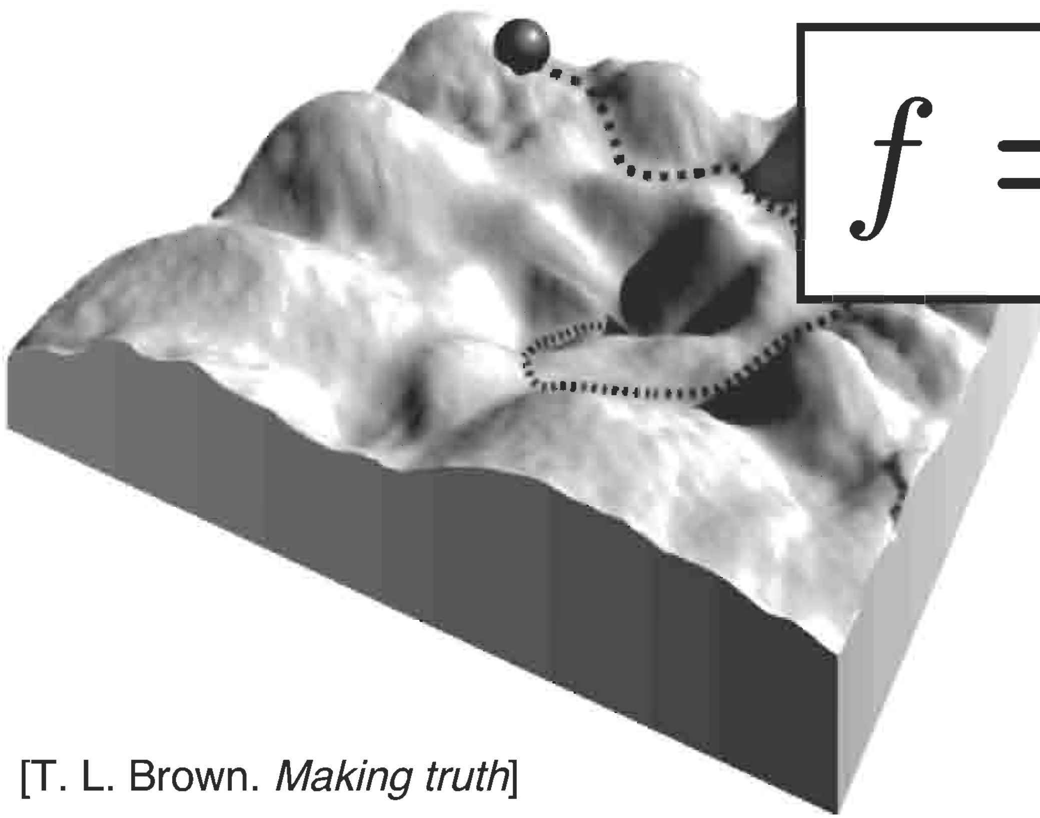


each config
maps to a poi
on energy
landscape

[T. L. Brown. *Making truth*]

Problem setup

Internal forces push “downhill”



$$f = -\nabla E$$

[T. L. Brown. *Making truth*]

Plates

Germain

Poisson

Navier

Thin plate energy

Germain's argument:

- bending energy must be a symmetric even function of principal curvatures

Thin plate energy

Germain's argument:

- bending energy must be a symmetric even function of principal curvatures

$$E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4}\int (\kappa_1 + \kappa_2)^2 dA$$

$$= \int H^2 dA$$

Thin plate energy

Poisson's linearization

- assuming small displacements, approximate curvature by second derivatives

$$E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4}\int(\kappa_1 + \kappa_2)^2 dA$$

$$E^{bend}_{lin} = \int(\Delta f)^2 dA$$

Thin plate energy

Navier’s equation

- to find minimizer for linearized energy, solve a partial differential eqn (PDE)

$$\Delta^2 f = 0$$

$$E_{lin}^{bend} = \int (\Delta f)^2 dA$$

Thin plate energy

Navier's equation

- to find minimizer for linearized energy solv

$$\partial_{uuuu} f + 2\partial_{uuvv} f + \partial_{vvvv} f$$

$$\Delta^2 f = 0$$

$$E_{lin}^{bend} = \int (\Delta f)^2 dA$$

Axiomatic approach

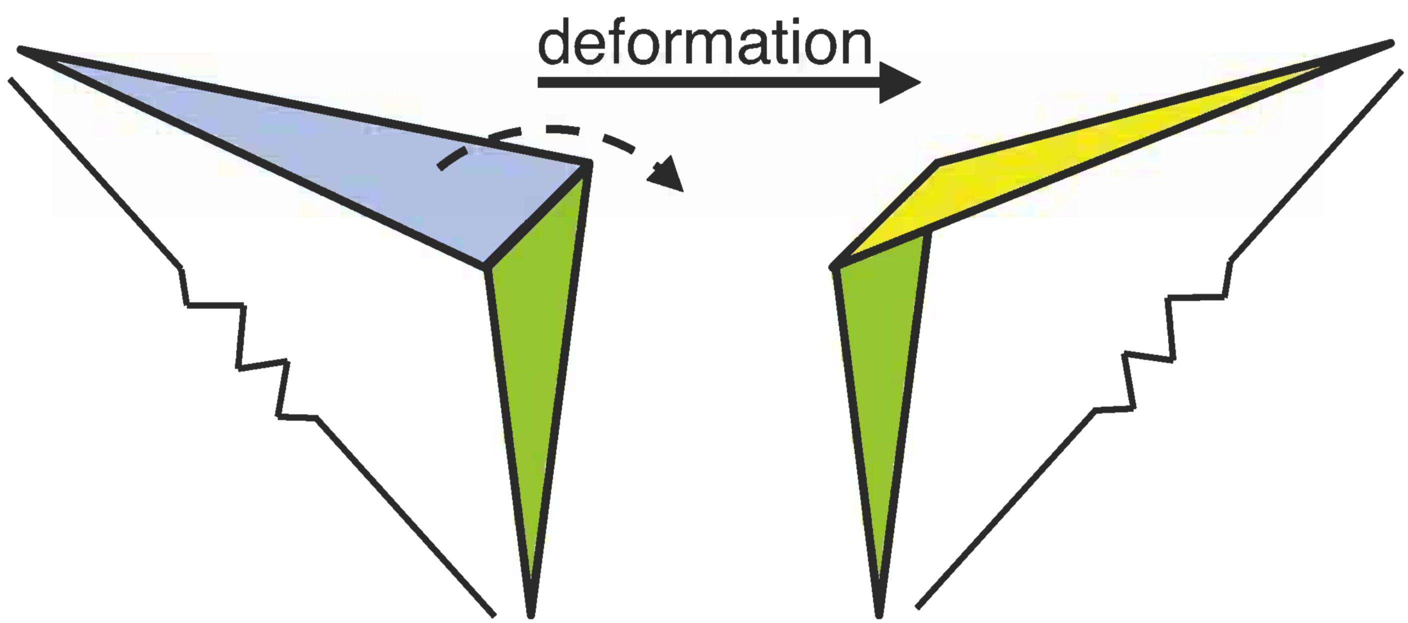
Energy should be:

- symmetric even func’n of principal curvatures
- extrinsic measure
- smooth w.r.t. change in shape
- invariant under rigid-body motion
- simple to compute
- easy to understand

What about masses and springs?

Diagonal springs don't work for shells.

- undeformed configuration is *curved*
- incorrect energy minima



Axiomatic “discrete shells”

“Simplest” answer to desiderata

$$\boxed{(H - H_0)^2}$$

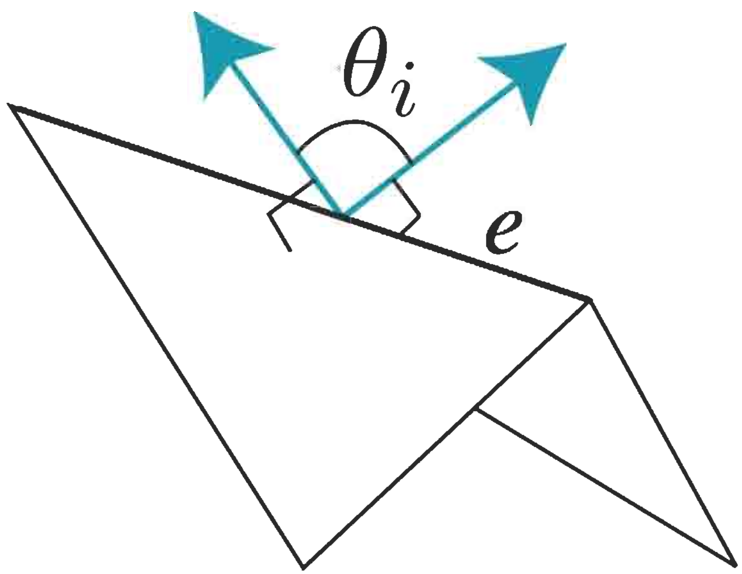
Derivation:

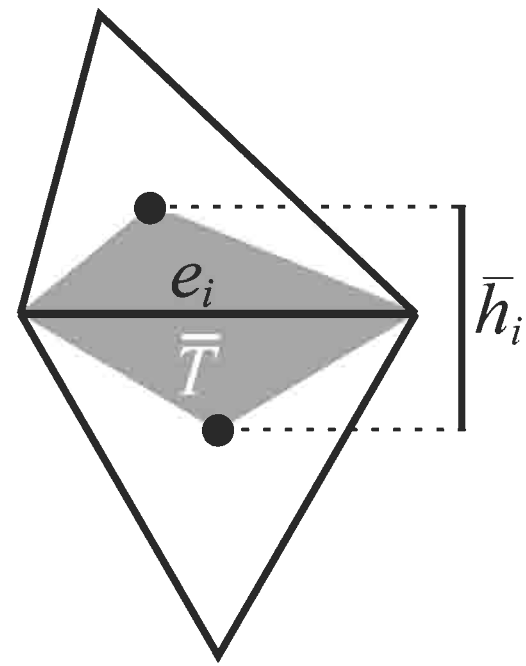
extrinsic change in *shape* operator

$$[\mathrm{Tr}(\varphi^* \mathsf{S}) - \mathrm{Tr}(\bar{\mathsf{S}})]^2$$

Computing discrete shells

Elastic energy = $\frac{K_B}{2}\sum_i(\theta_i - \bar{\theta}_i)^2\frac{\|\bar{e}_i\|}{\bar{h}_i}$

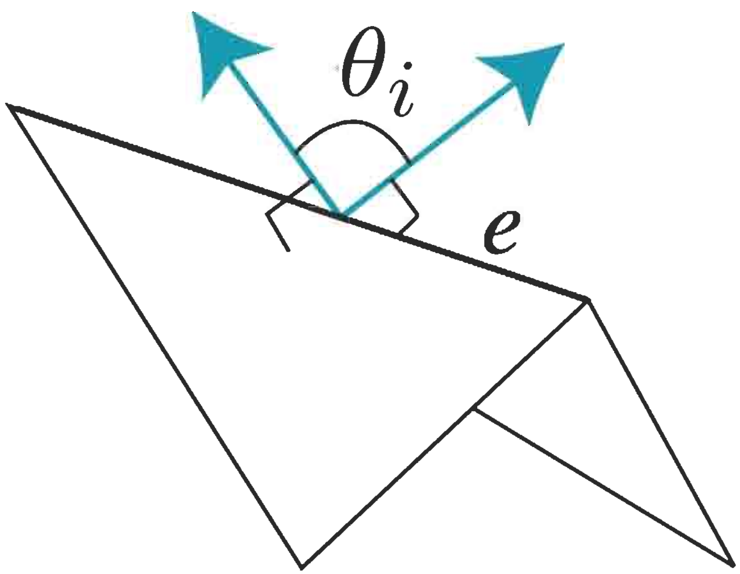


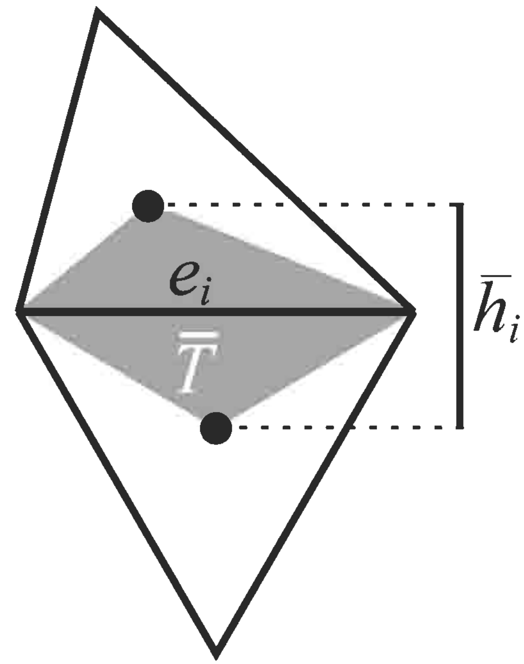


Computing discrete shells

material coef.

$$\text{Elastic energy} = \frac{K_B}{2}\sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\|\bar{e}_i\|}{\bar{h}_i}$$

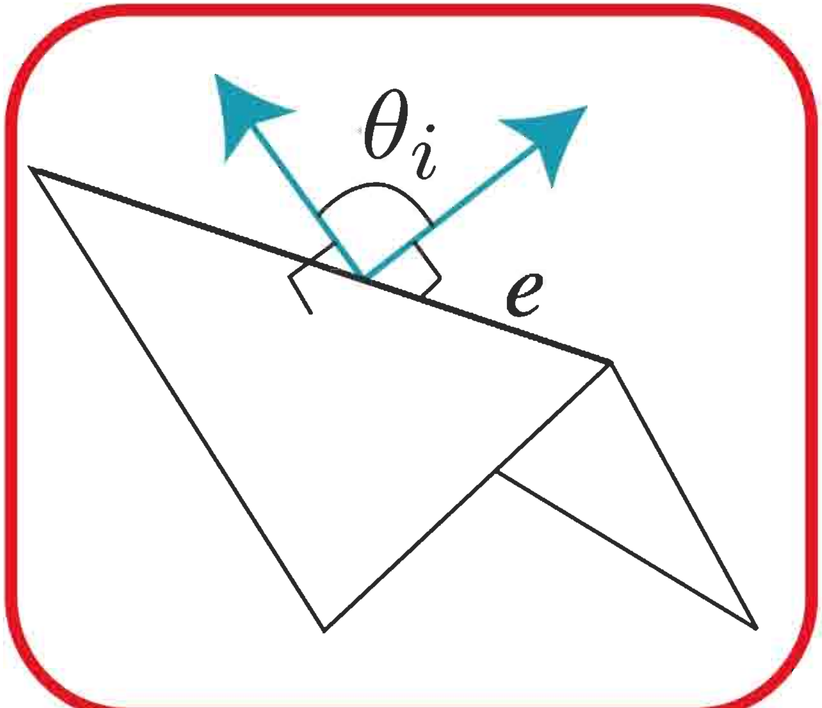


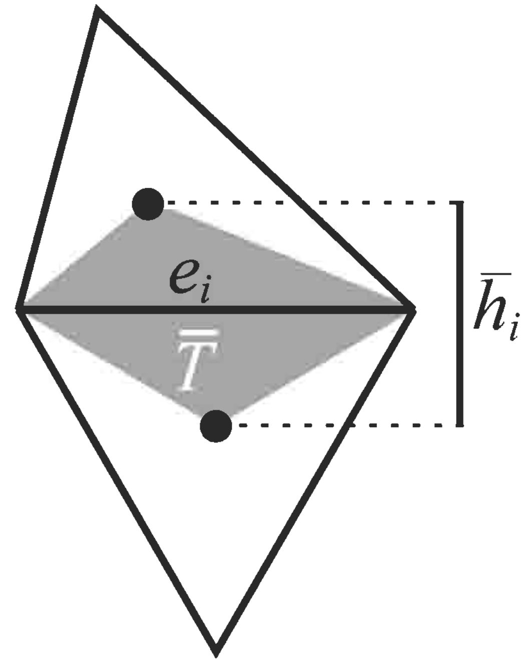


Computing discrete shells

change in normal curvature

Elastic energy = $\frac{K_B}{2}\sum_i(\theta_i - \bar{\theta}_i)^2\frac{\|\bar{e}_i\|}{\bar{h}_i}$

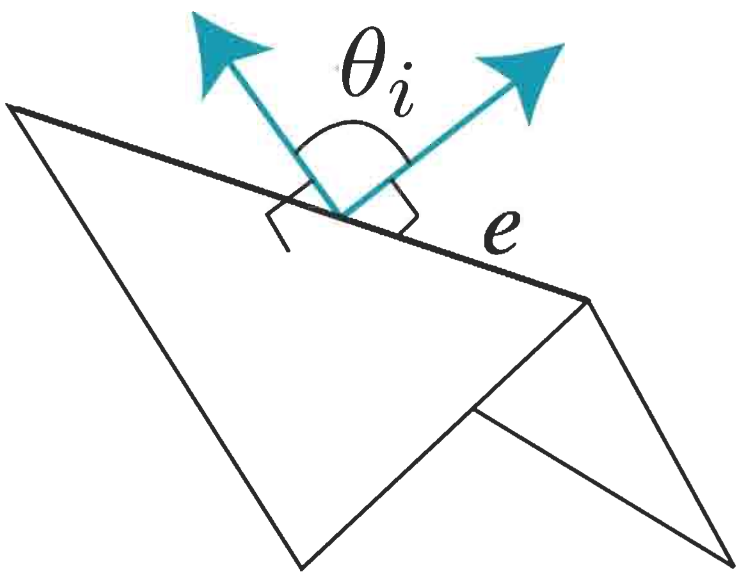


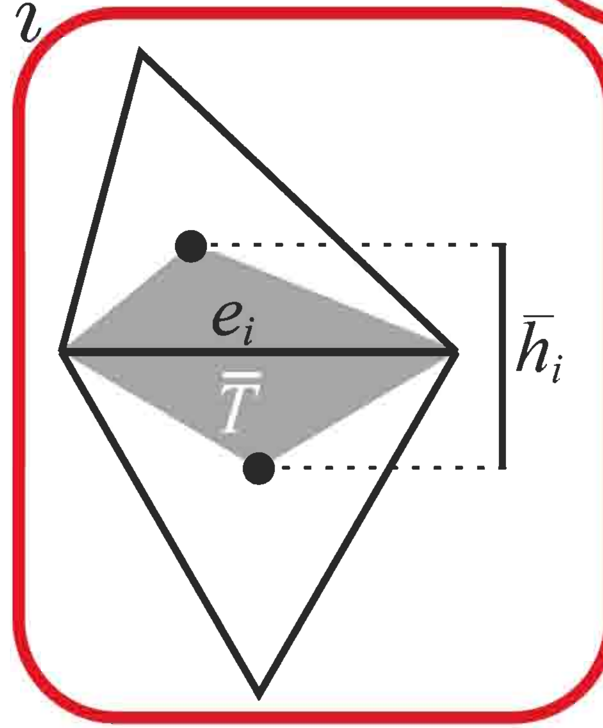


Computing discrete shells

compensate for nonuniform sampl

$$\text{Elastic energy} = \frac{K_B}{2}\sum_i(\theta_i - \bar{\theta}_i)^2\frac{\|\bar{e}_i\|}{\bar{h}_i}$$





Computing discrete shells

Elastic energy = $\frac{K_B}{2}\sum_i(\theta_i - \bar{\theta}_i)^2\frac{\|\bar{e}_i\|}{\bar{h}_i}$

Gradient gives forces:

$$f_k = K_B\sum_i \frac{\|\bar{e}_i\|}{\bar{h}_i}(\bar{\theta}_i - \theta_i)\nabla_{\mathbf{x}_k}\theta_i$$

Upgrade your cloth simulator

Have a cloth simulator handy?

- reuse all the existing code
- retrofit the bending term
- precompute undeformed quantities offline

$$f_k = K_B \sum_i \frac{\|\bar{e}_i\|}{\bar{h}_i} (\bar{\theta}_i - \theta_i) \nabla_{\mathbf{x}_k} \theta_i$$

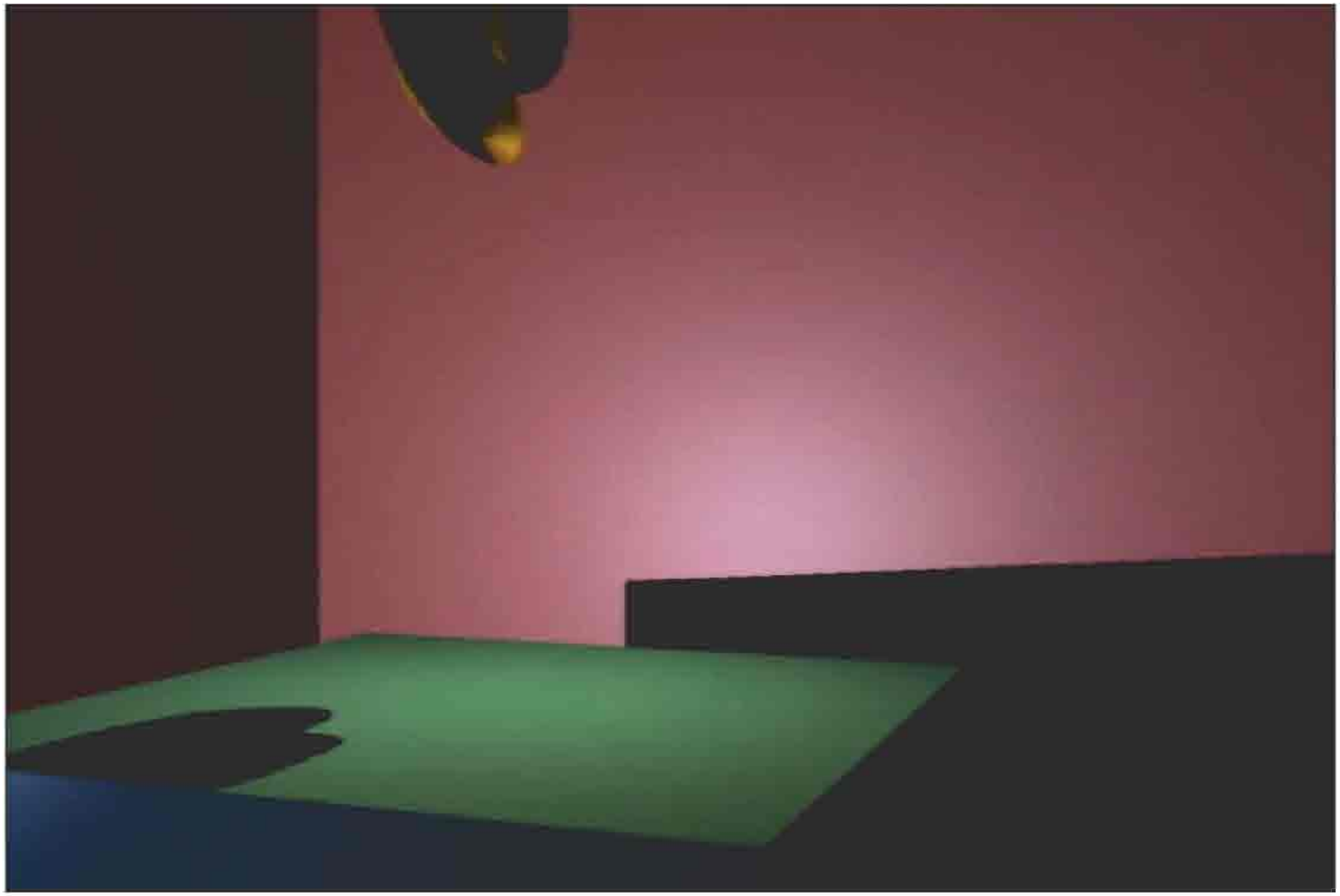
Upgrade your cloth simulator

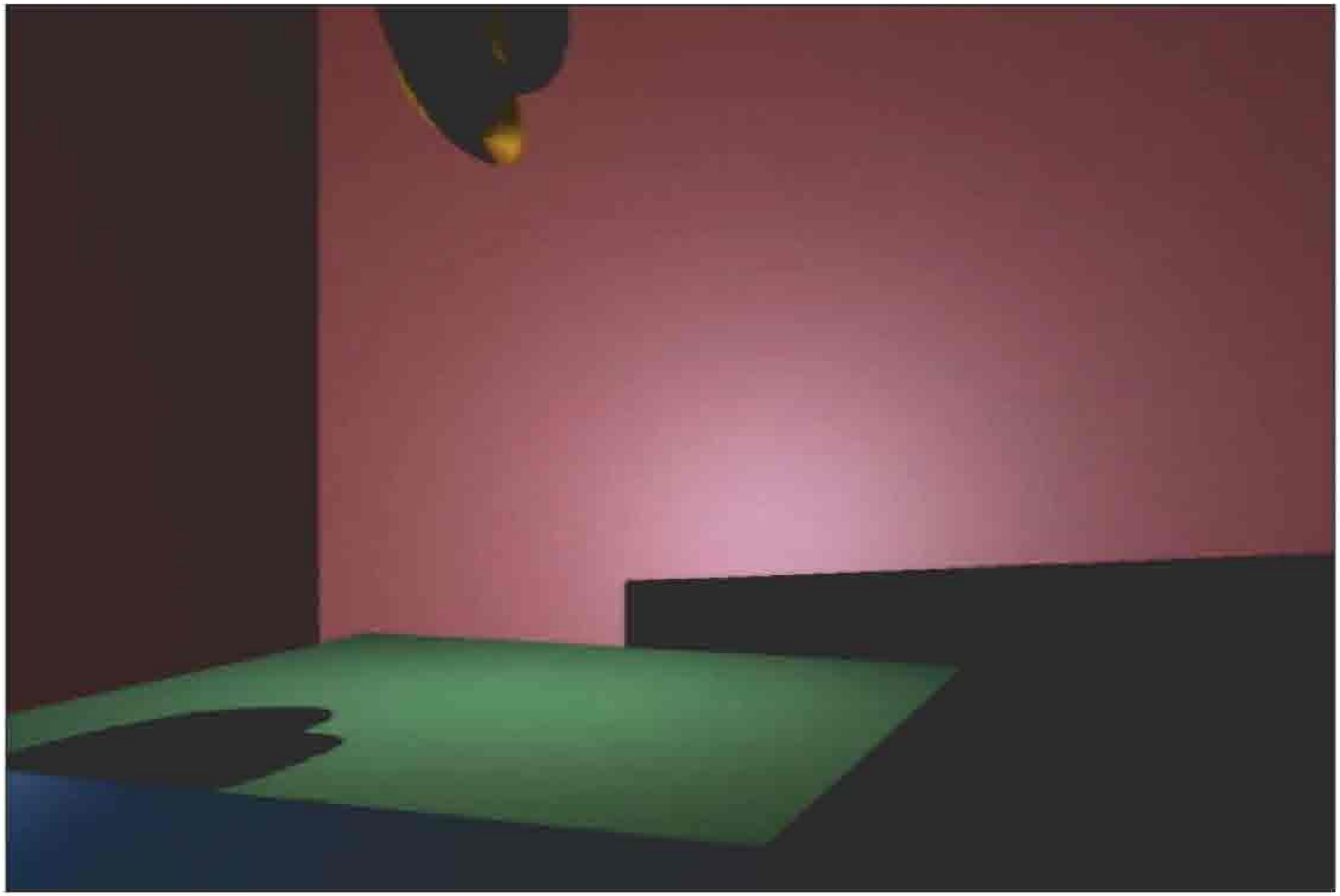
Have a cloth simulator handy?

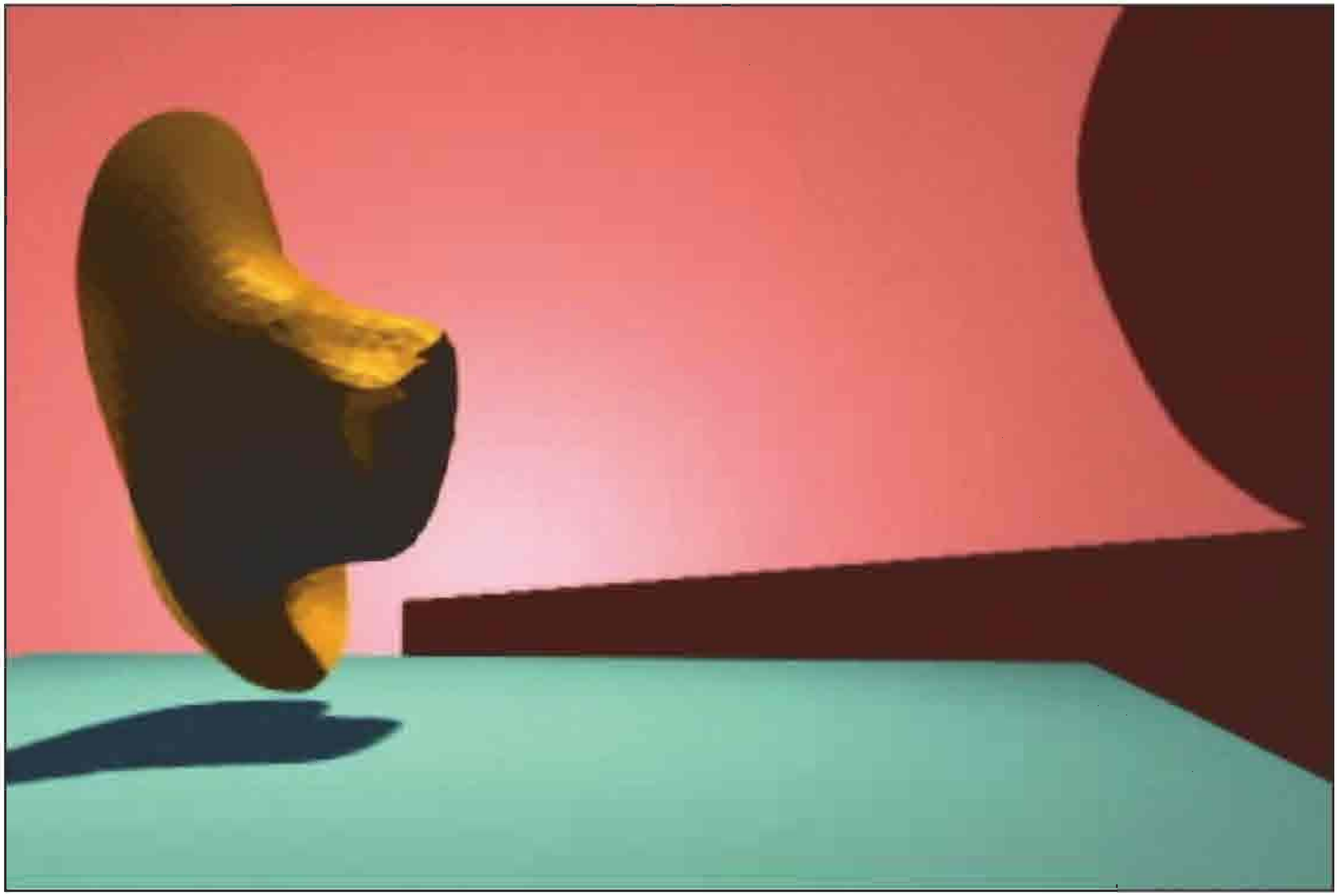
- reuse all the existing code
- retrofit the bending term
- precompute undeformed quantities offline

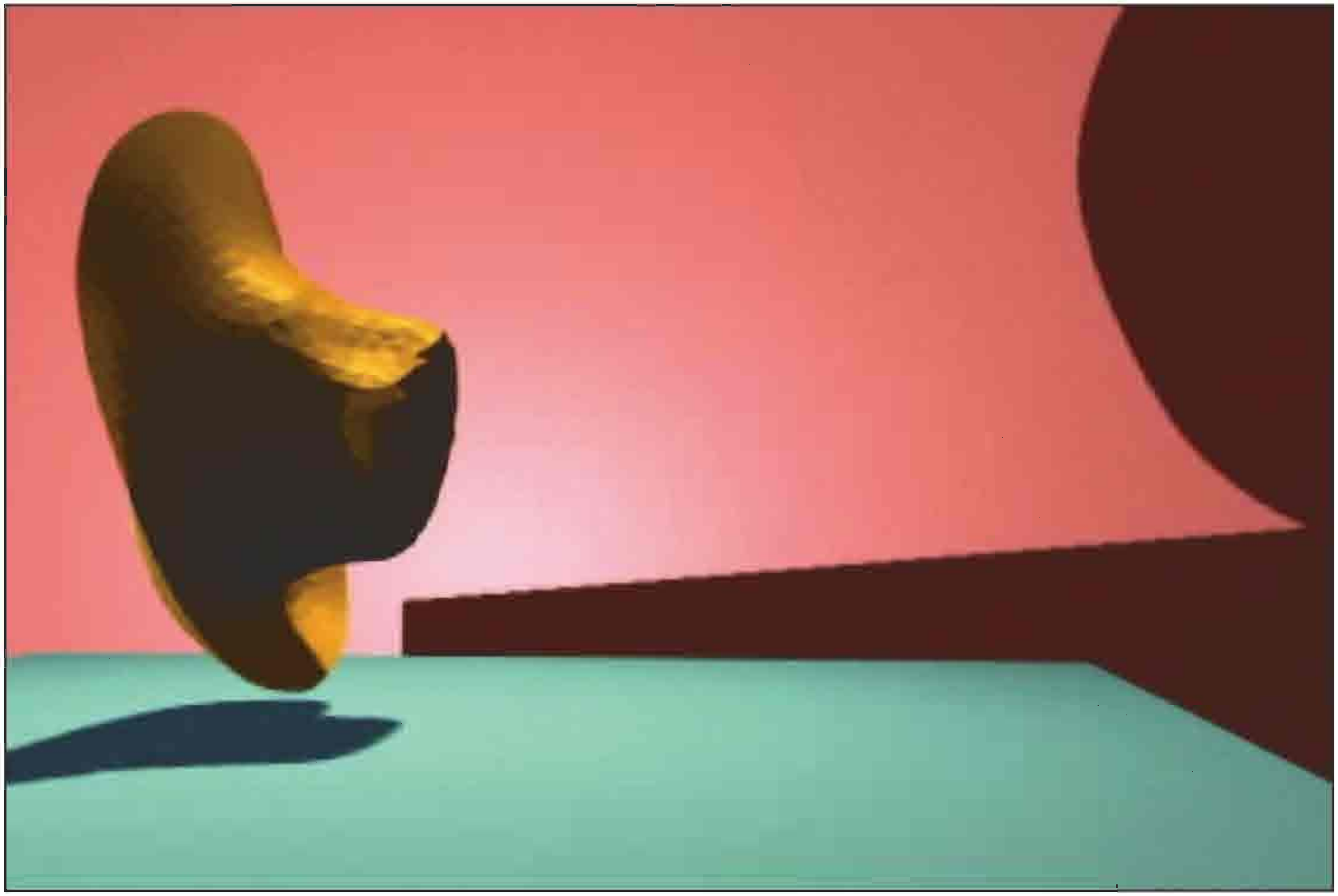
new code   reuse code

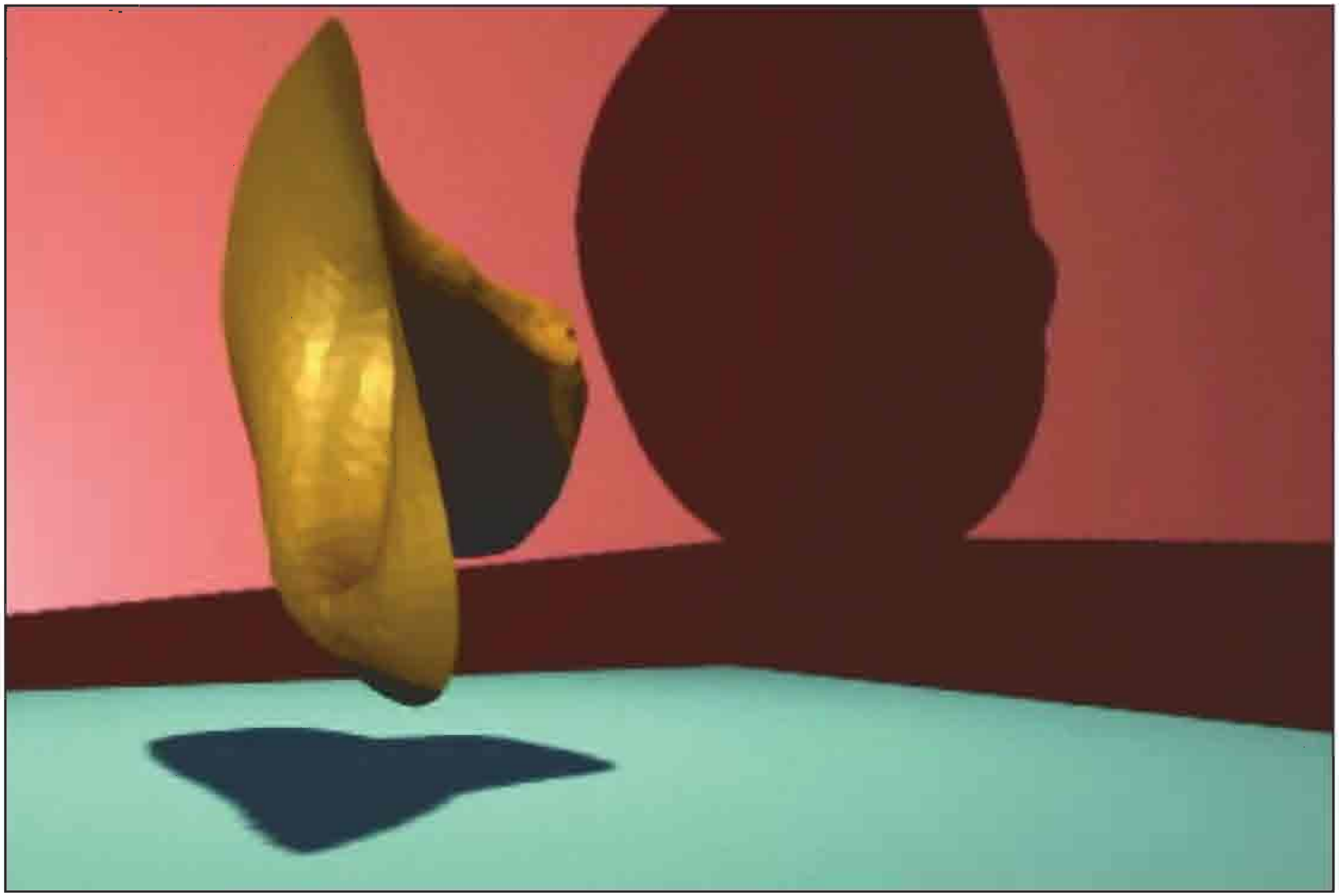
$$f_k = K_B \sum_i \frac{\|\bar{e}_i\|}{\bar{h}_i}(\bar{\theta}_i - \theta_i)\nabla_{\mathbf{x}_k}\theta_i$$

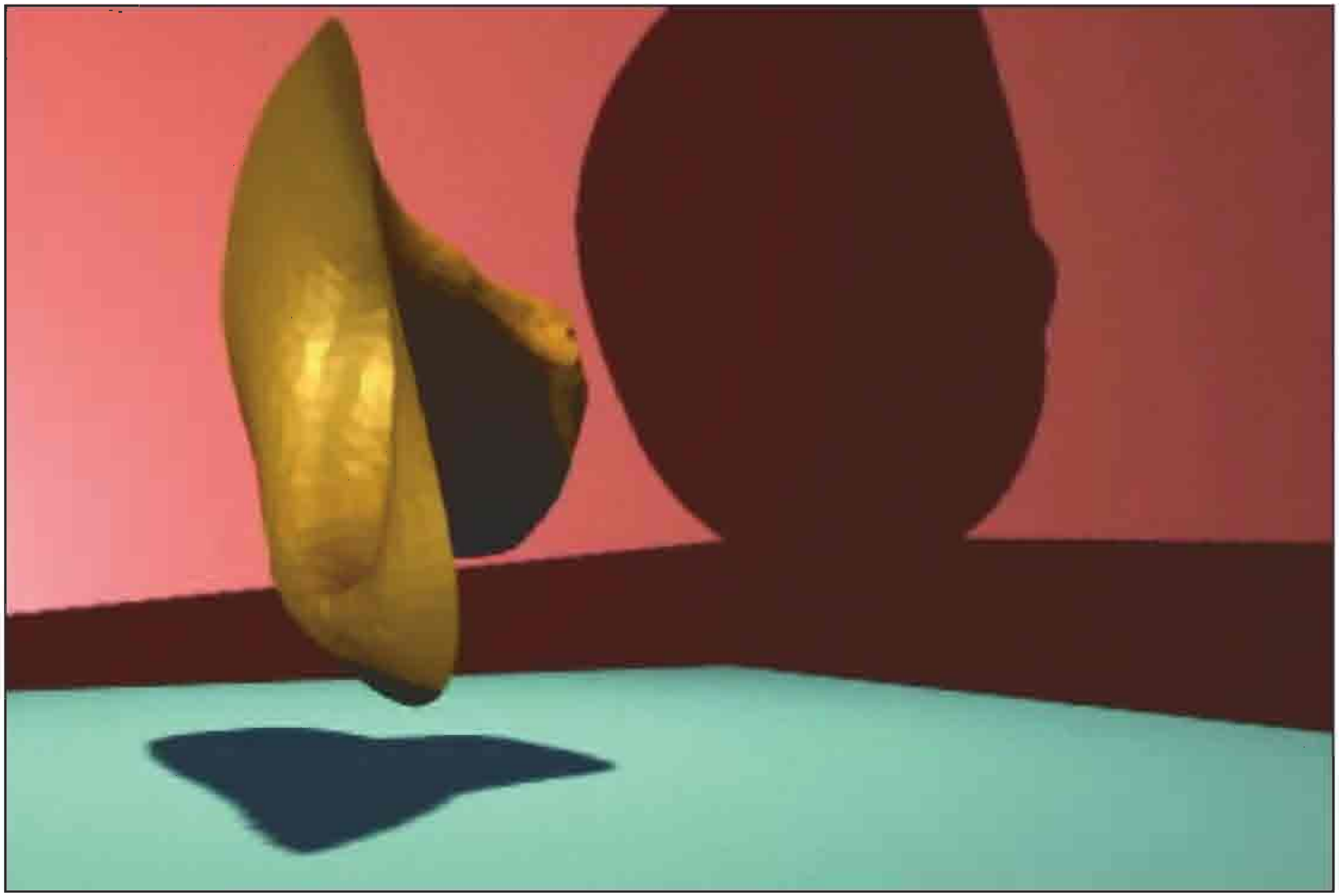












Modeling Paper

Paper sheet

- curled
- creased
- pinned

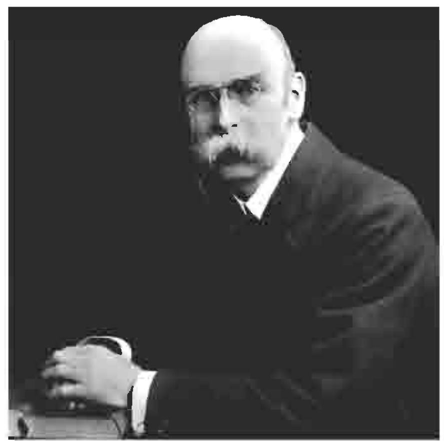
Are we done?

Discrete shells is nice and simple.

What's next?

Thin shell theory

Kirchhoff



Love

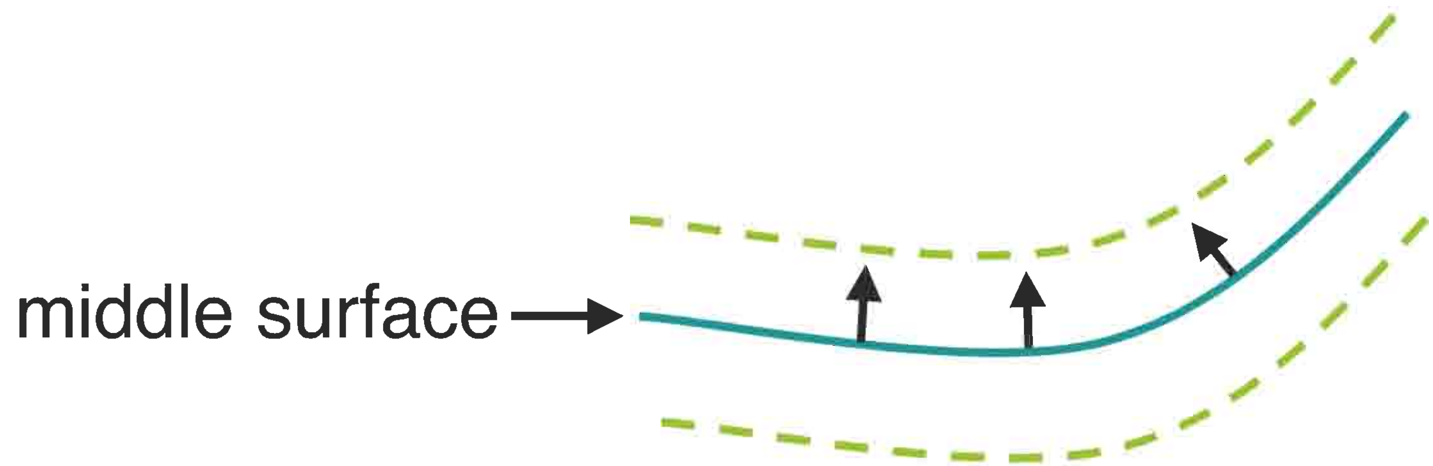
Karman

Koiter

Shell geometry

Shell representation:

middle surface + normal offset



Stored energy

Step 1: strain

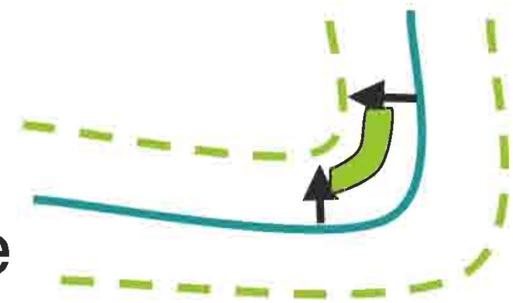
- deformation of small volume

Step 2: energy density

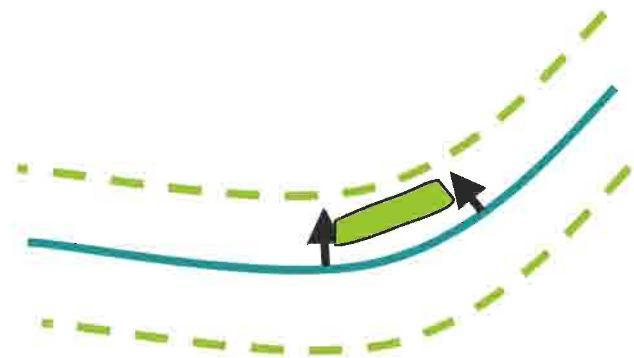
- compute work
- constitutive model

Step 3: integrate

- over shell thickness
- over middle surface



undeformed

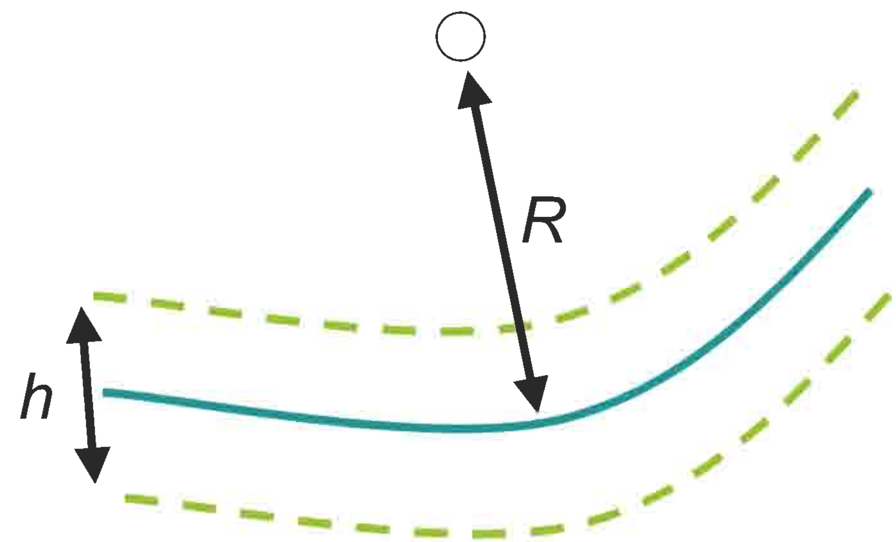


deformed

Assumptions

Thin shell

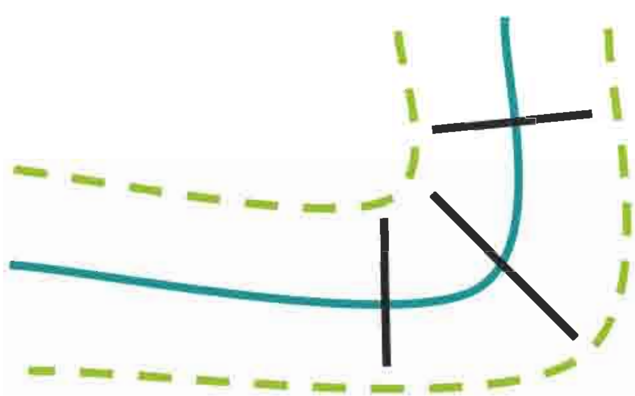
- thickness much less than radius of curvature



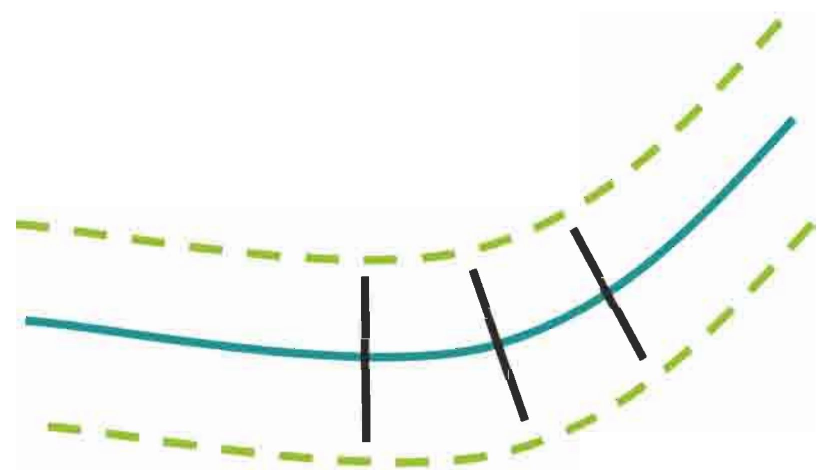
Assumptions

Kirchhoff-Love

- normal lines deform to normal lines



undeformed

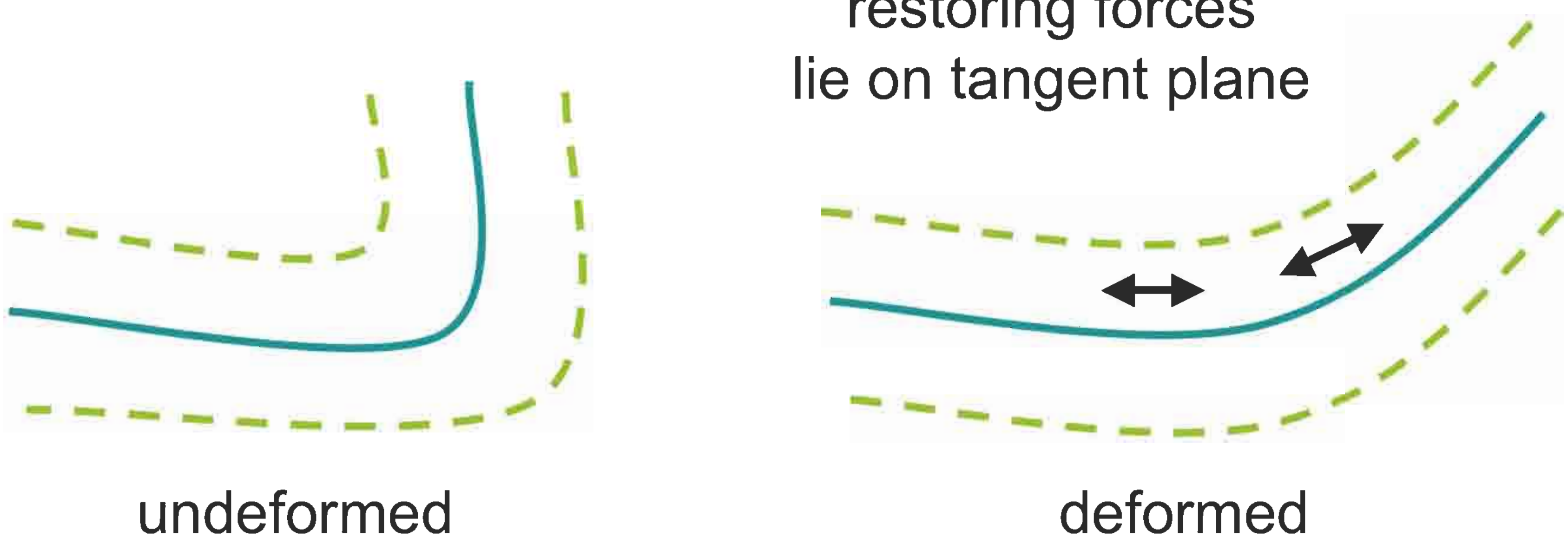


deformed

Assumptions

Planar stress

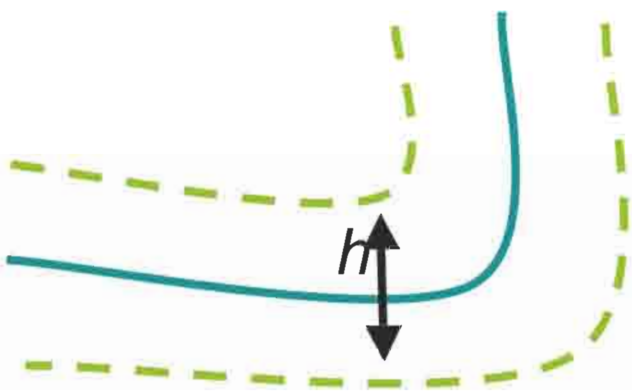
- neglect stress in normal direction



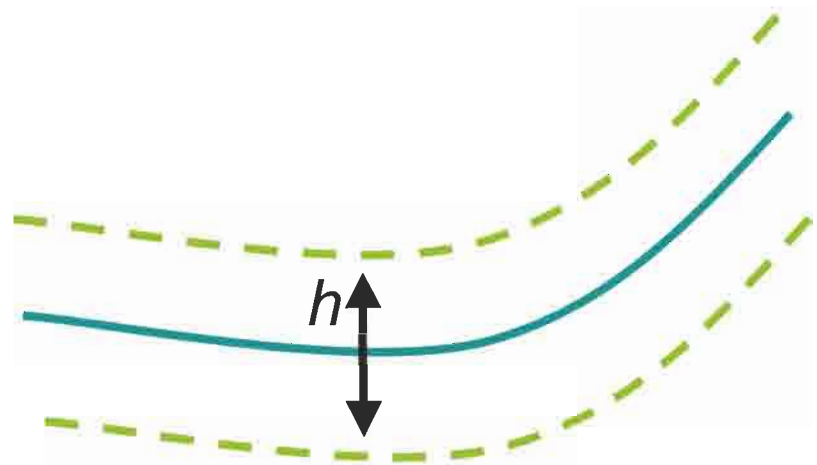
Assumptions

Normal inextensibility

- distance preserved along normal lines



undeformed



deformed

Assumptions

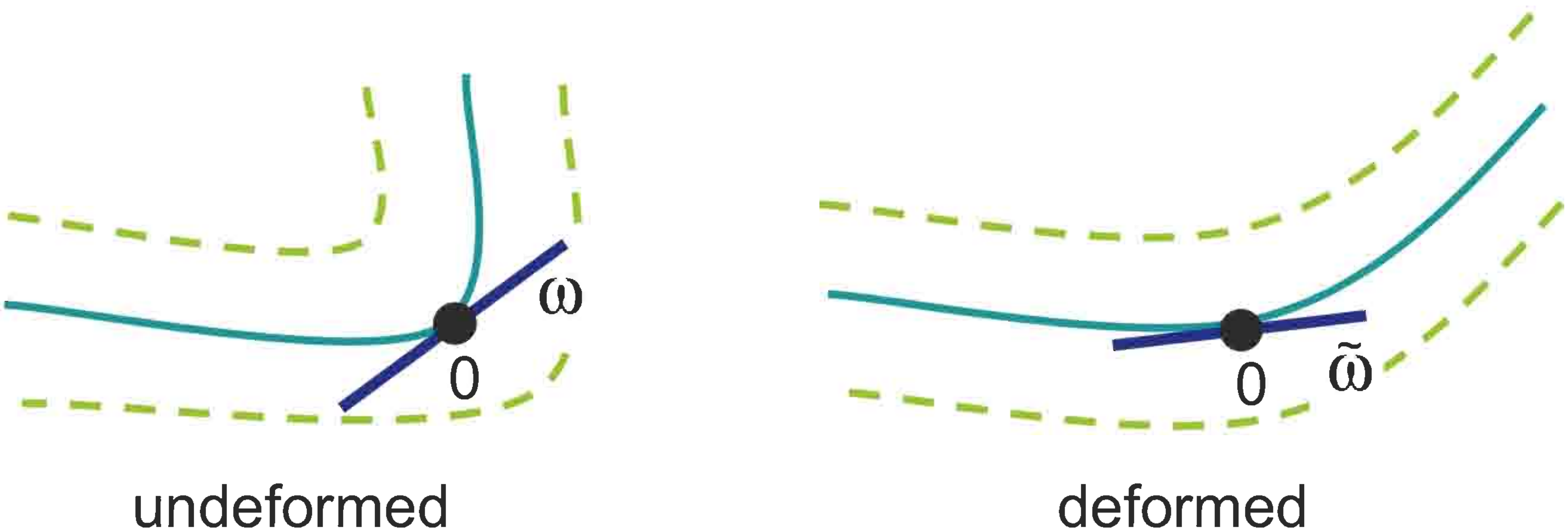
Small strains

- all strains are small
  - strains in glass: O(0.0001)
  - strains in paper: O(0.01)
- deflections may be large

Shell geometry

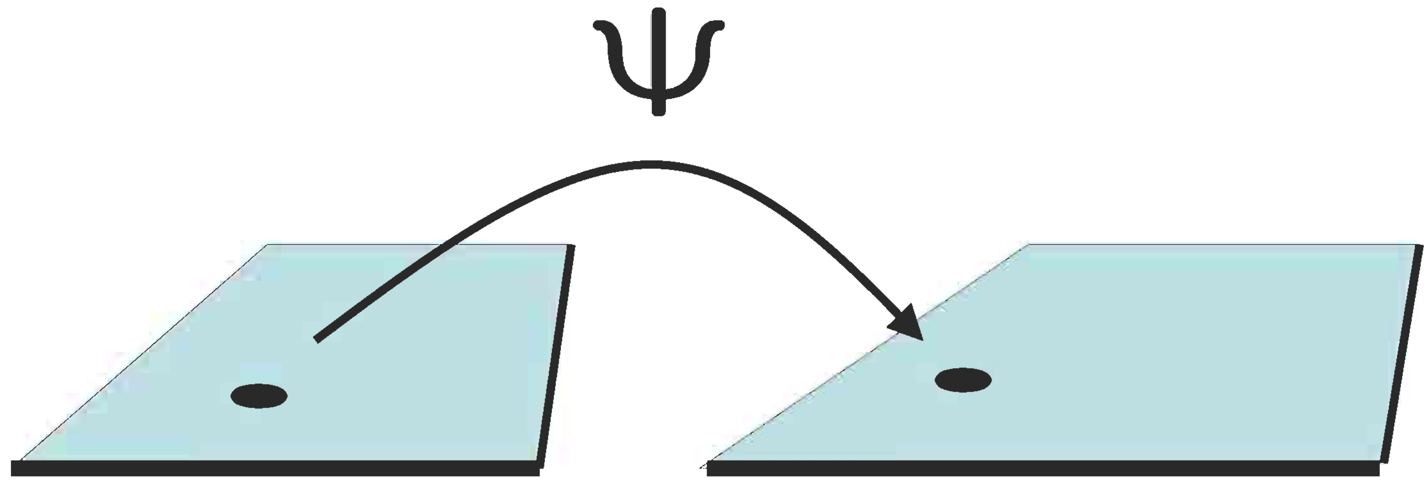
Locally description

- consider small neighborhood
- global parameterization *not* required



Deformation mapping

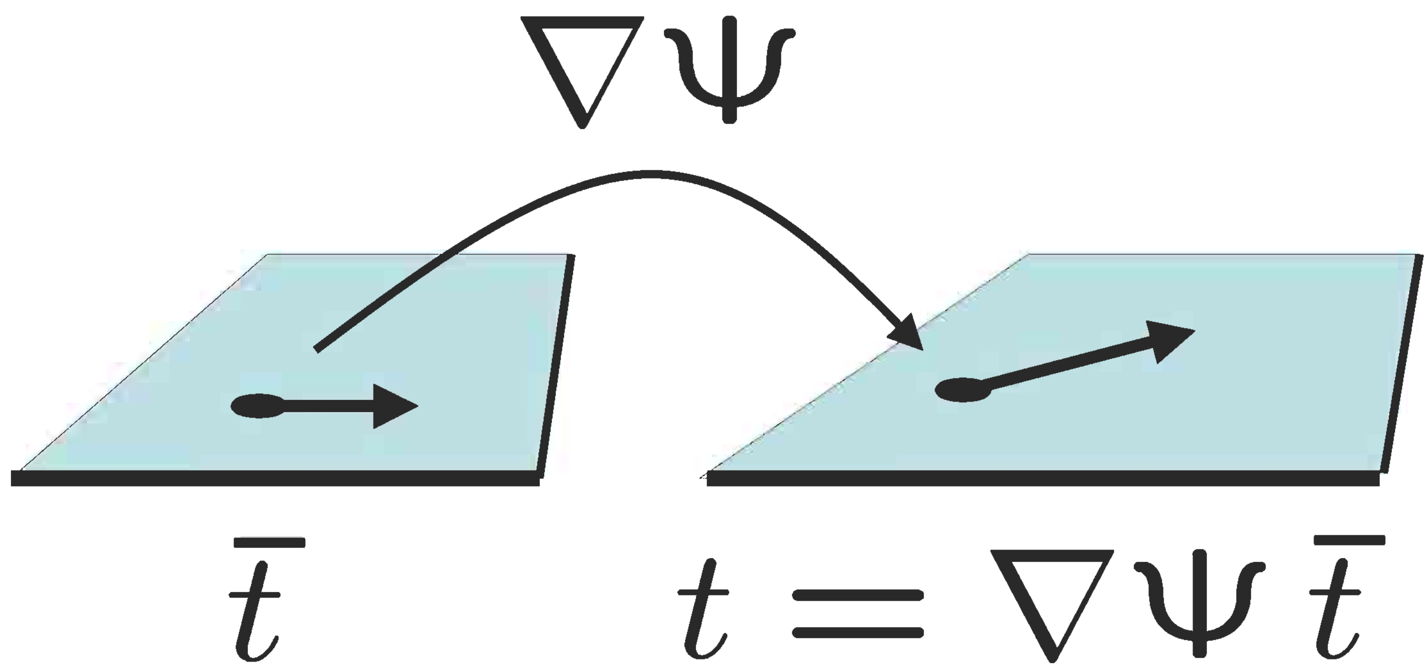
Maps material point on tangent plane
from undeformed to deformed config.



$\bar{x}$ $\quad$ $x = \Psi(\bar{x})$

Deformation gradient

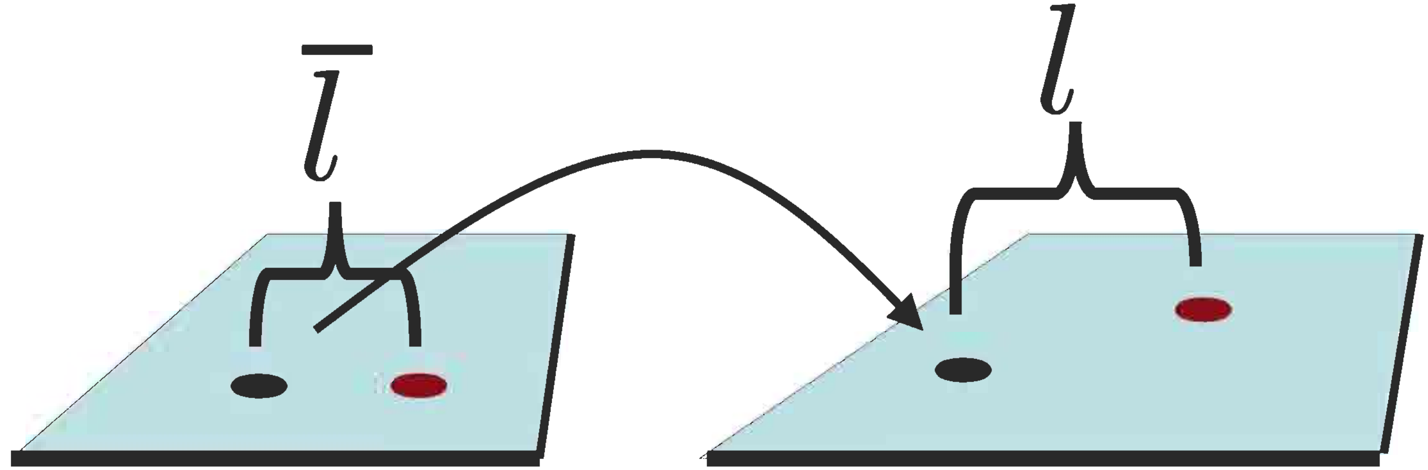
Maps tangent vector
from undeformed to deformed config.



Strain

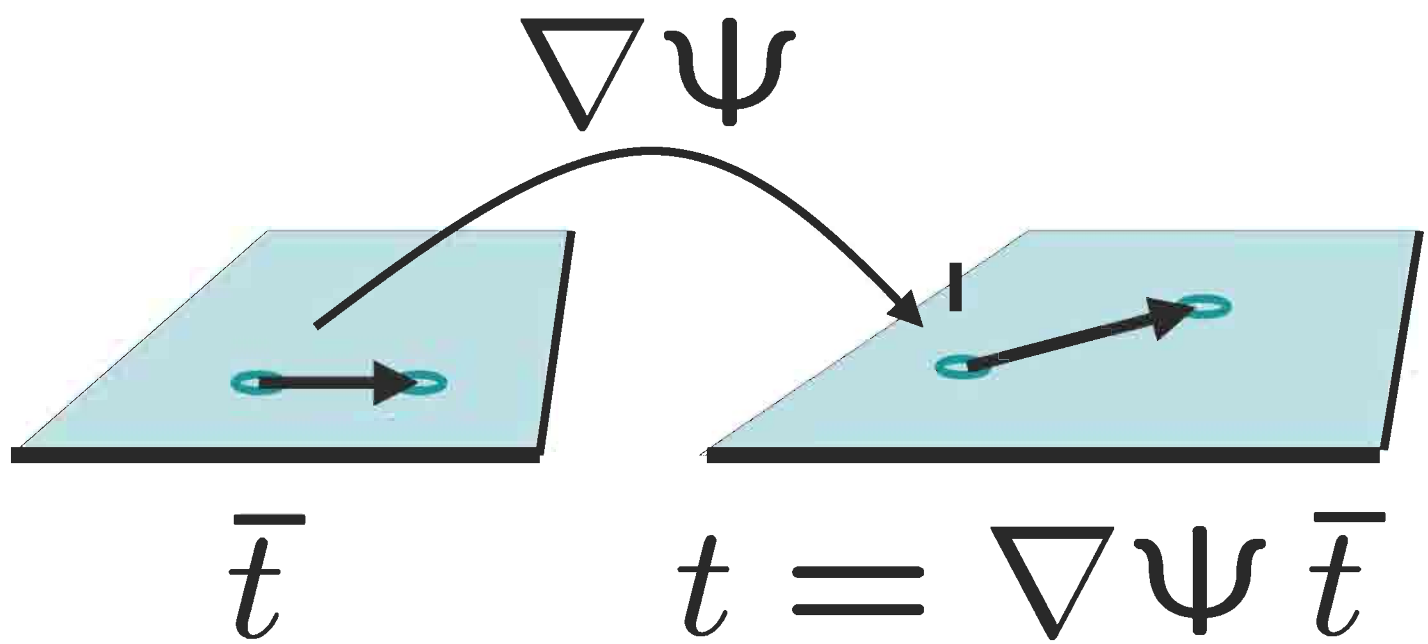
Change in squared length between two nearby points

$$l^2 - \bar{l}^2$$



Strain

Change in squared length of tangent vector



Strain

Change in squared length of tangent vector

$$|t|^2 - |\bar{t}|^2 = t^T t - \bar{t}^T \bar{t}$$

Strain

Change in squared length of tangent vector

$$|t|^2 - |\bar{t}|^2 = \underbrace{t^T}\underbrace{t} - \bar{t}^T\bar{t}$$

$$t = \nabla\Psi\,\bar{t}$$

$$t^T = \bar{t}^T(\nabla\Psi)^T$$

Strain

Change in squared length of tangent vector

$$\bar{t}^T(\nabla\Psi)^T(\nabla\Psi)\bar{t} - \bar{t}^T\bar{t}$$

Strain

Change in squared length of tangent vector

$$\bar{t}^T (\nabla \Psi)^T (\nabla \Psi) \bar{t} - \bar{t}^T \bar{t}$$

$$\bar{t}^T ((\nabla \Psi)^T (\nabla \Psi) - I) \bar{t}$$

Strain

Change in squared length of tangent vector

$$\bar{t}^T (\nabla\Psi)^T (\nabla\Psi)\bar{t} - \bar{t}^T\bar{t}$$

$$\bar{t}^T \left((\nabla\Psi)^T (\nabla\Psi) - I \right)\bar{t}$$

Strain

Change in squared length of tangent vector

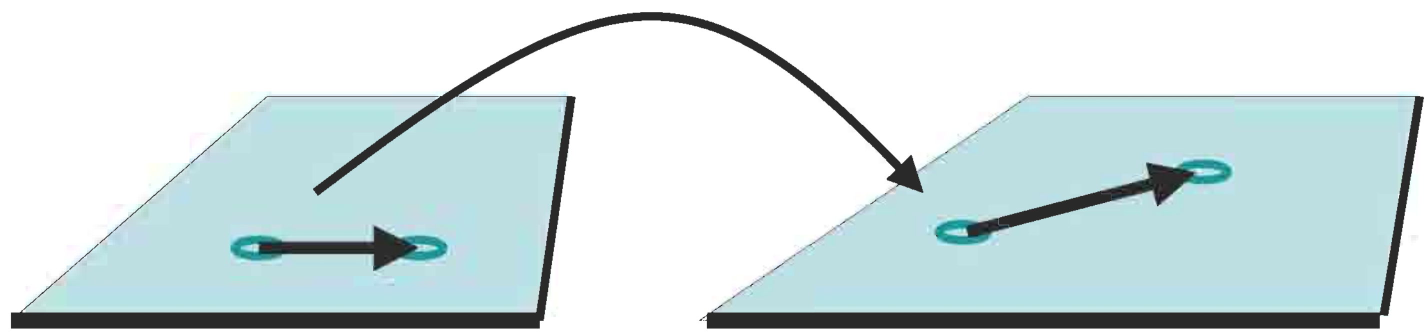
$$E_m$$

$$\boxed{(\nabla\Psi)^T(\nabla\Psi) - I}$$

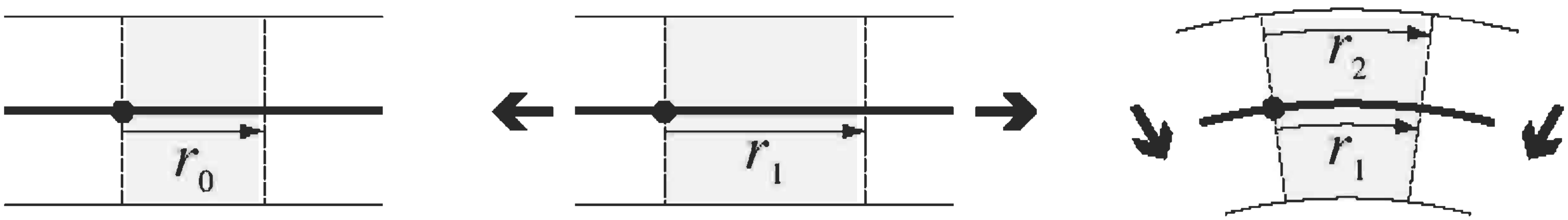
Strain

Change in squared length of tangent vector

$$\bar{t}^T E_m \bar{t}$$



Thought experiment



begin at rest,

...stretch the shell,

...then bend the shell

$$r_1 = (I + E_m)r_0$$

$$r_2 = (I + zE_c)r_1$$

$$\approx (I + E_m + zE_c)$$

Membrane + bending strain

$$E(z) = E_m + z\Delta\Lambda = E_m + zE_c$$

Surface Energy Density

Energy formulation - bending term

$$\frac{Yh^3}{24(1-\nu^2)}\left((1-\nu)\mathrm{Tr}(E_c^2)+\nu(\mathrm{Tr}E_c)^2\right)$$

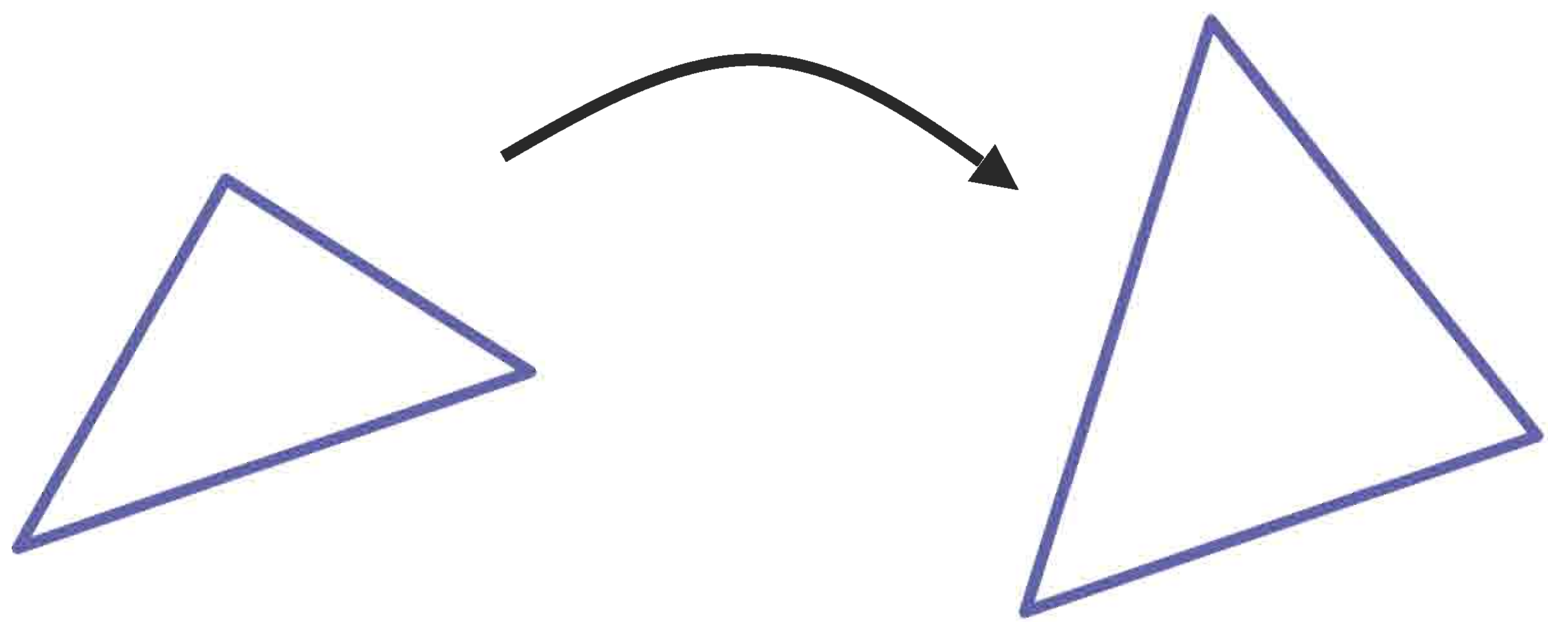
- geometric interpretation

$$(1+\nu)\Delta H^2+(1-\nu)(\Delta A^2+4A\tilde{A}\sin^2\beta)$$

  - change in mean curvature
  - change in curvature direction

Planar strain

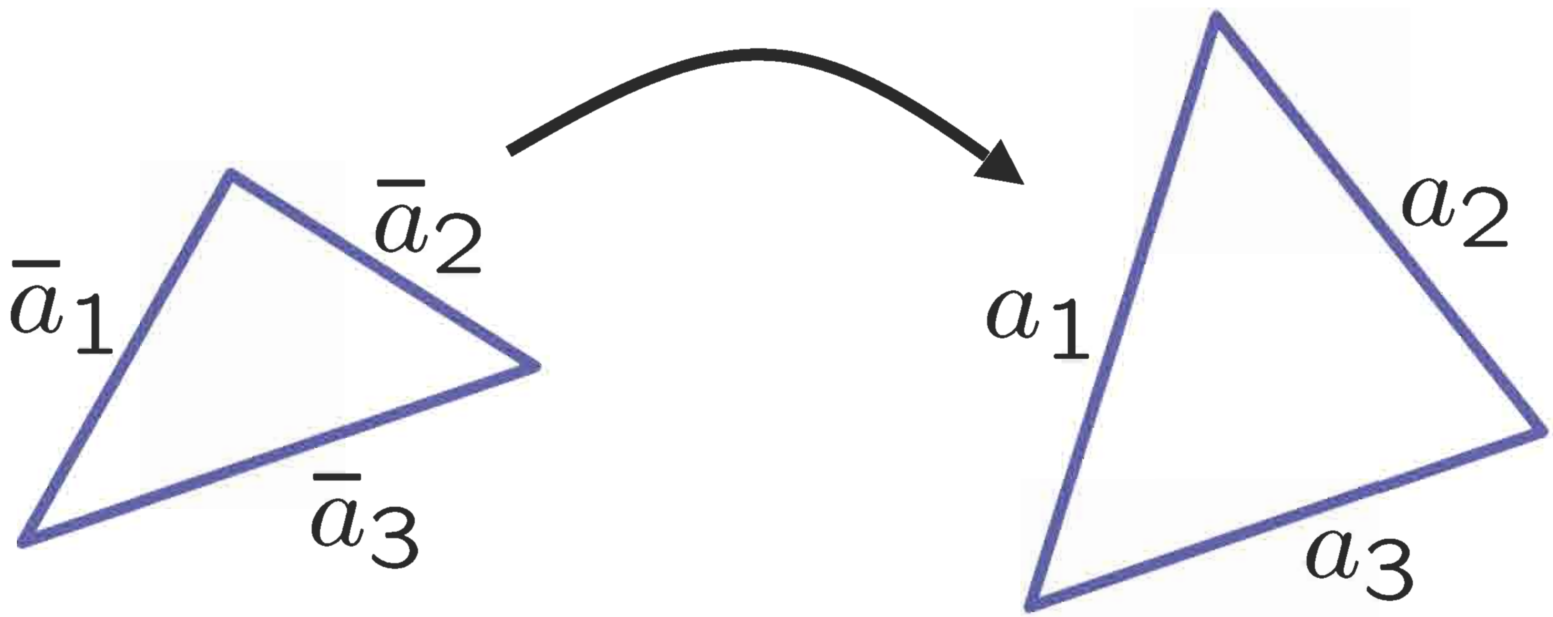
Consider a single deformed triangle…



Planar strain

Function only of change in edge lengths!

- (thank you Hadwiger!)

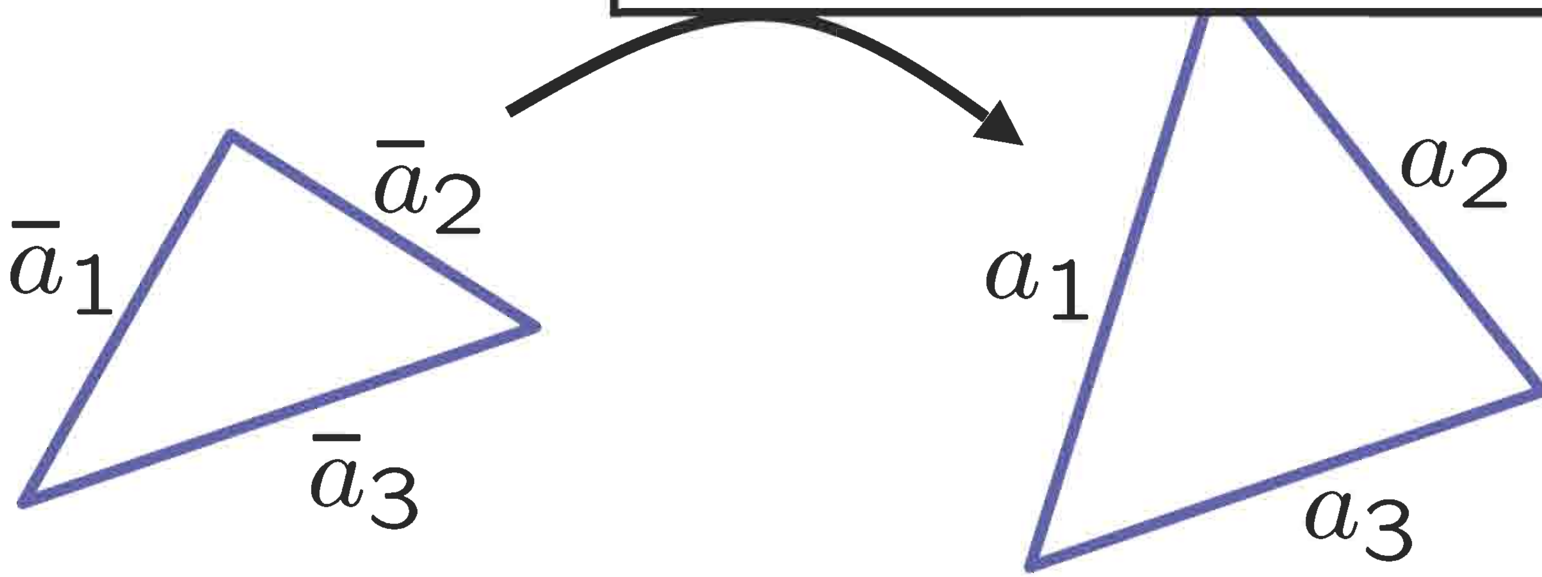


Planar strain

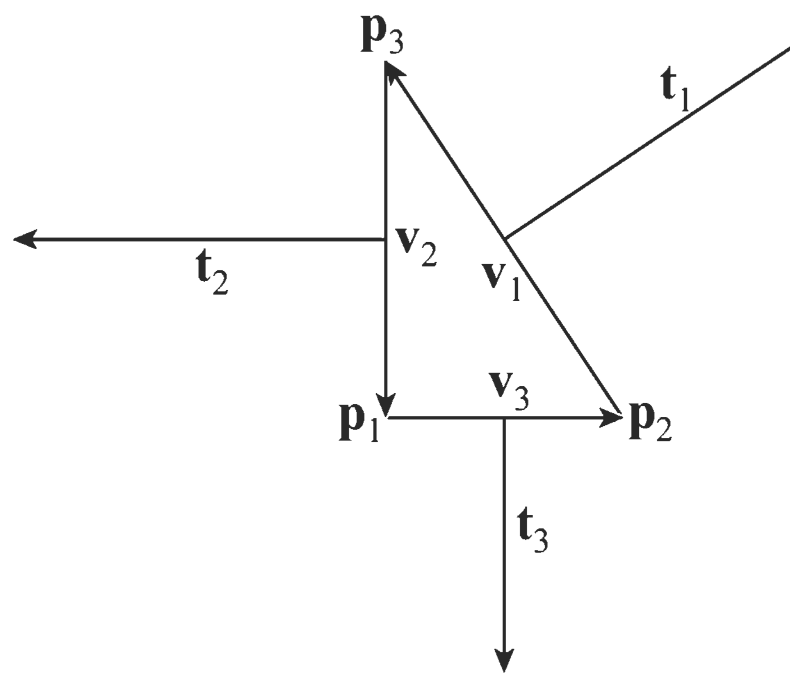
Function only of c

- (thank you Ha

$$
\begin{aligned}
s_1 &= a_1^2 - \bar{a}_1^2 \\
s_2 &= a_2^2 - \bar{a}_2^2 \\
s_3 &= a_3^2 - \bar{a}_3^2
\end{aligned}
$$



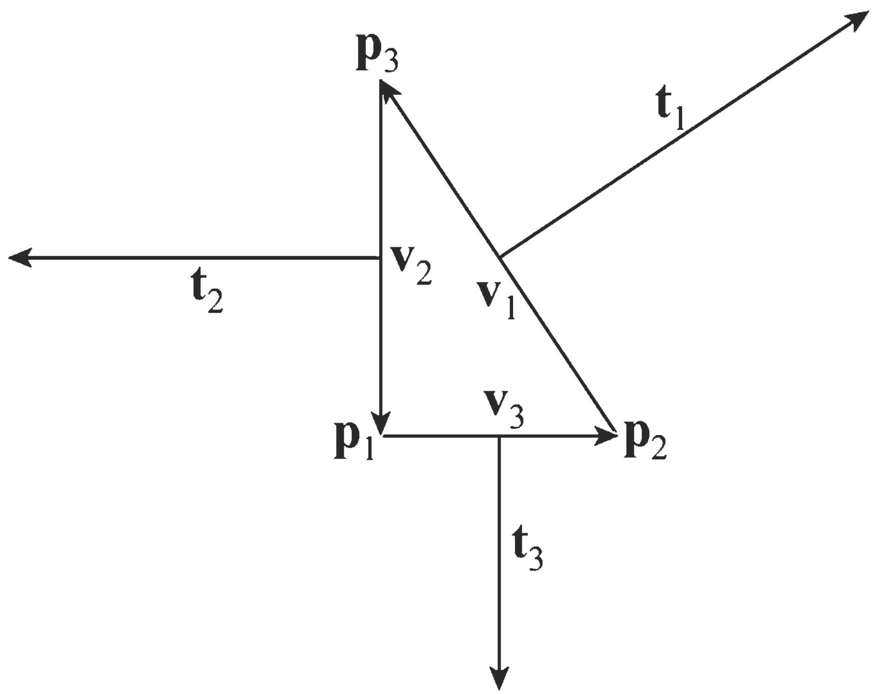
Planar strain

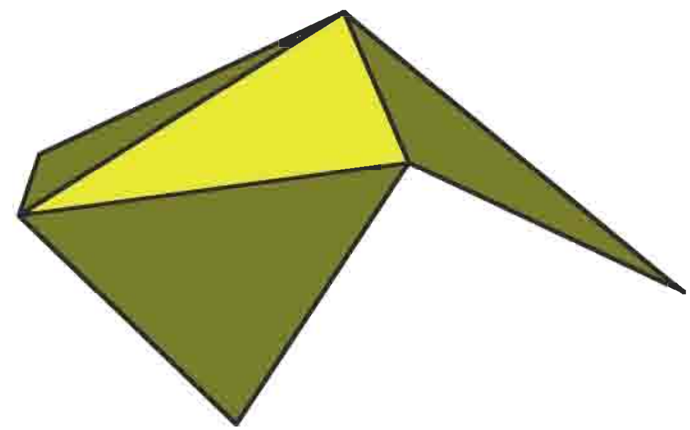


$$
\begin{aligned}
s_1 &= a_1^2 - \bar{a}_1^2 \\
s_2 &= a_2^2 - \bar{a}_2^2 \\
s_3 &= a_3^2 - \bar{a}_3^2
\end{aligned}
$$

$$E_m = \frac{1}{8A^2} \sum_{i=1}^{3} s_i \left(t_j \otimes t_k + t_k \otimes t_j \right)$$

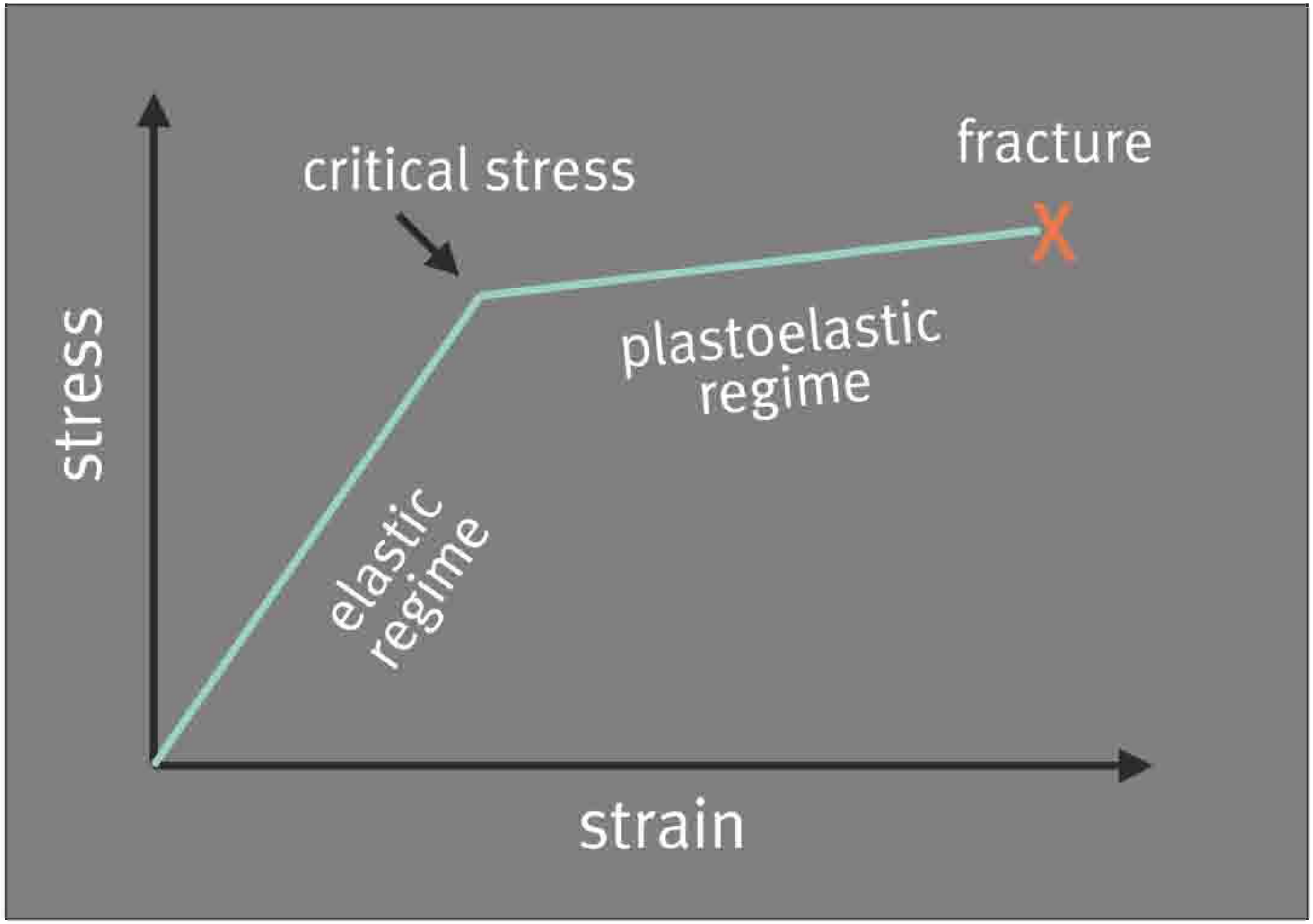
Discrete bending strain





$$E_b = \frac{1}{2A}\sum_{i=1}^{3}(\theta - \bar{\theta})\frac{t_i \otimes t_i}{a_i}$$

Material Failure

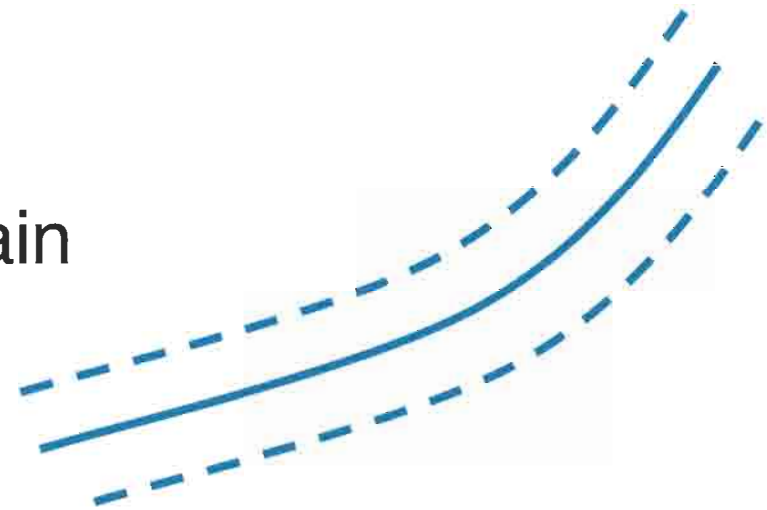


Material Failure

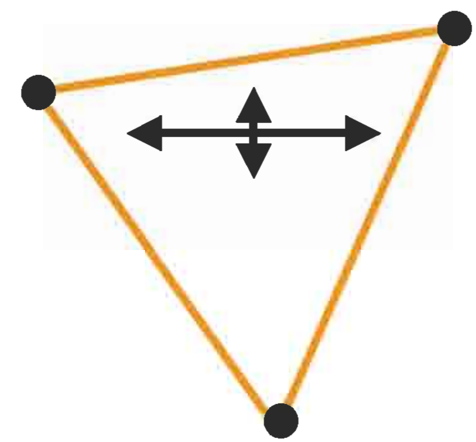
Principal strains

- compute maximal strain

$$E_m \pm hE_c$$



- eigenvalue > threshold
    - split against eigenvector



Results: plastic deformation

Falling Tube

Results: plastic deformation

Falling Tube

Results: Fracture

Results: Fracture

Results: Fracture

Results: Fracture

Results: Fracture

Lightbulbs

Results: Fracture

Lightbulbs

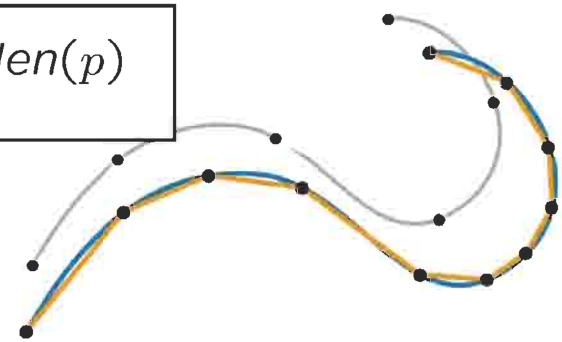
What about convergence?

With sufficient refinement, does the discrete energy agree with the continuous energy?

The length of a continuous curve

…or take limit over refinement sequence.

$$\lim_{h \to 0} len(p)$$



Work in progress…

Convergence… why it matters

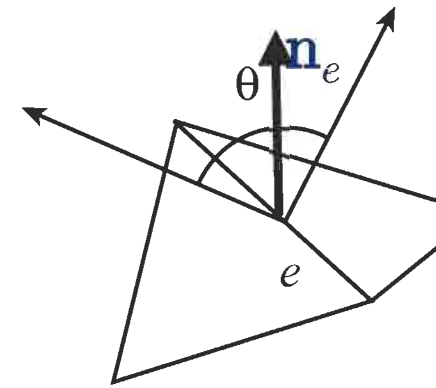
Maybe small systematic error is ok?

- mesh independence for fine meshes
- adaptive refinement
  - requires error criteria
  - ∴ limit must be well-defined
- for simulation
  - can use physical parameters
- for variational modeling
  - canonical surface definition

Discrete curvature measures

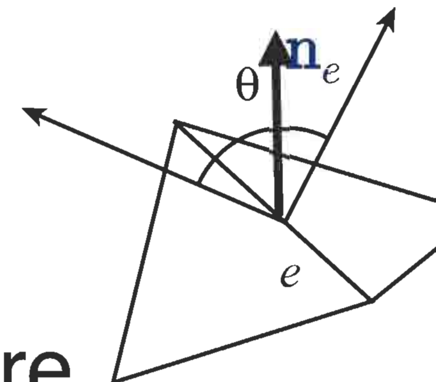
Basic building block $\mathbf{H}_e = \mathbf{n}_e |\mathbf{e}| f(\frac{\theta}{2})$

- total curvature normal



Discrete curvature measures

Basic building block $\mathbf{H}_e = \mathbf{n}_e |\mathbf{e}| f(\frac{\theta}{2})$



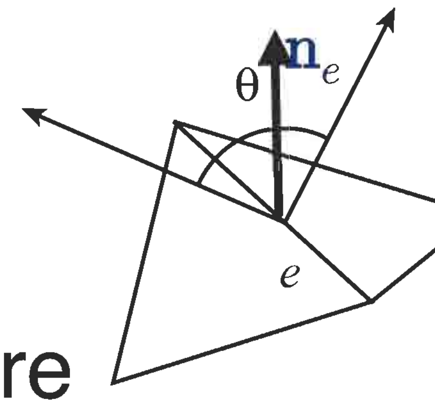
- total curvature normal

Variations of total mean curvature

- per edge per triangle per vertex

Discrete curvature measures

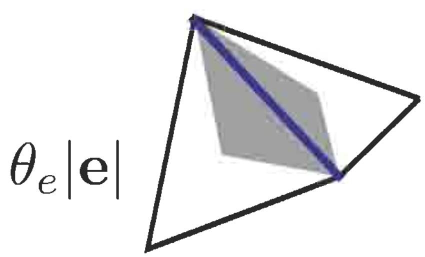
Basic building block $\mathbf{H}_e = \mathbf{n}_e |\mathbf{e}| f(\frac{\theta}{2})$



- total curvature normal

Variations of total mean curvature

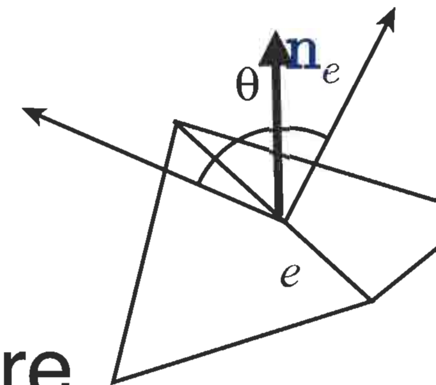
- per edge   per triangle   per vertex



discrete shells,
hinge energy

Discrete curvature measures

Basic building block $\mathbf{H}_e = \mathbf{n}_e |\mathbf{e}| f(\frac{\theta}{2})$

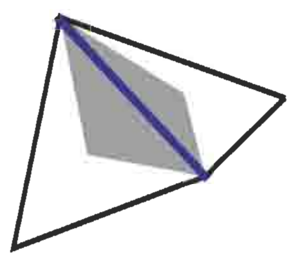


- total curvature normal

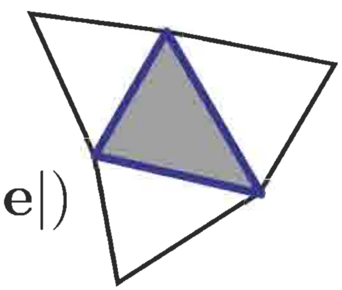
Variations of total mean curvature

- per edge   per triangle   per vertex

$\theta_e |\mathbf{e}|$



$\frac{1}{2}(\sum_e \theta_e |\mathbf{e}|)$

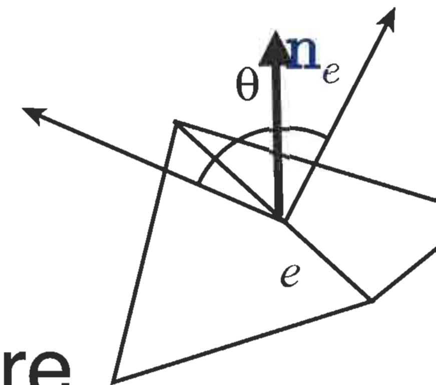


discrete shells,
hinge energy

triangle-averaged

Discrete curvature measures

Basic building block $\mathbf{H}_e = \mathbf{n}_e |\mathbf{e}| f(\frac{\theta}{2})$

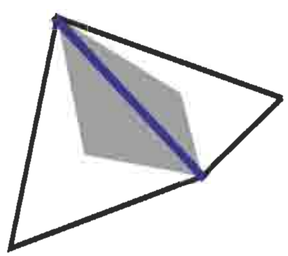


- total curvature normal

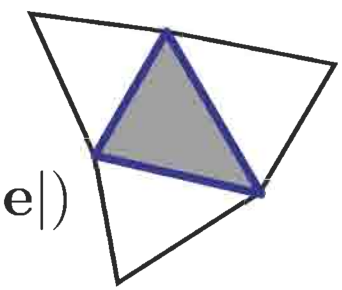
Variations of total mean curvature

- per edge   per triangle   per vertex

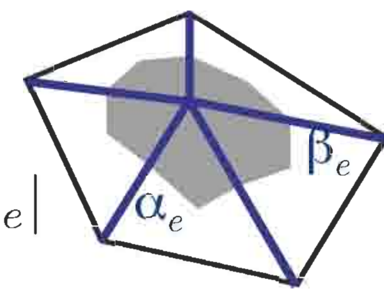
$\theta_e |\mathbf{e}|$



$\frac{1}{2}(\sum_e \theta_e |\mathbf{e}|)$



$\frac{1}{2}|\sum_e \mathbf{H}_e|$



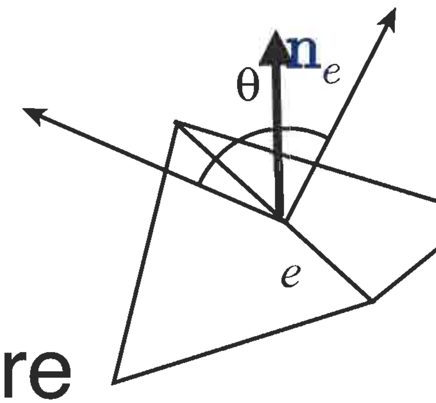
discrete shells,
hinge energy

triangle-averaged

vertex-averaged

Discrete curvature measures

Basic building block $\mathbf{H}_e = \mathbf{n}_e |\mathbf{e}| f(\frac{\theta}{2})$

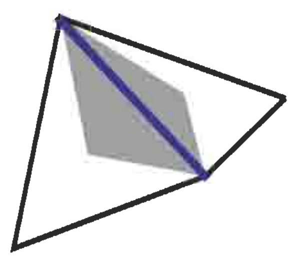


- total curvature normal

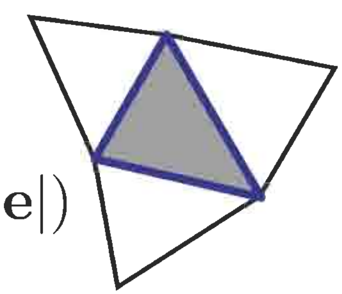
Variations of total mean curvature

- per edge per triangle per vertex

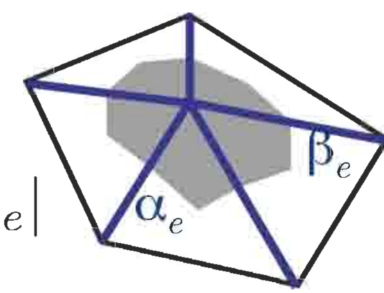
$\theta_e |\mathbf{e}|$



$\frac{1}{2}(\sum_e \theta_e |\mathbf{e}|)$



$\frac{1}{2}|\sum_e \mathbf{H}_e|$



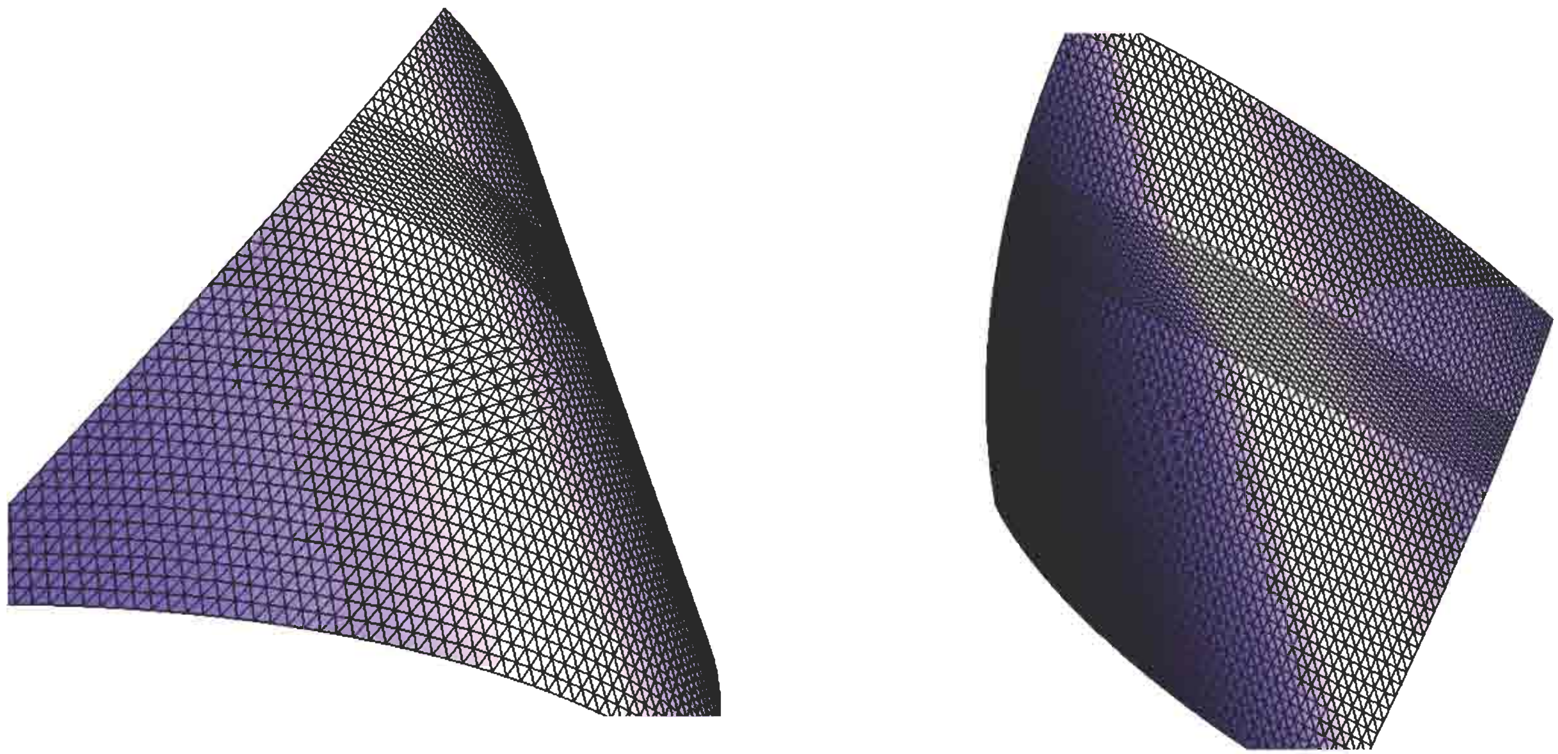
discrete shells, hinge energy

triangle-averaged

vertex-averaged

$$\frac{1}{2}|\sum_e (\cot \alpha_e + \cot \beta_e)\mathbf{e}|$$

Test problem

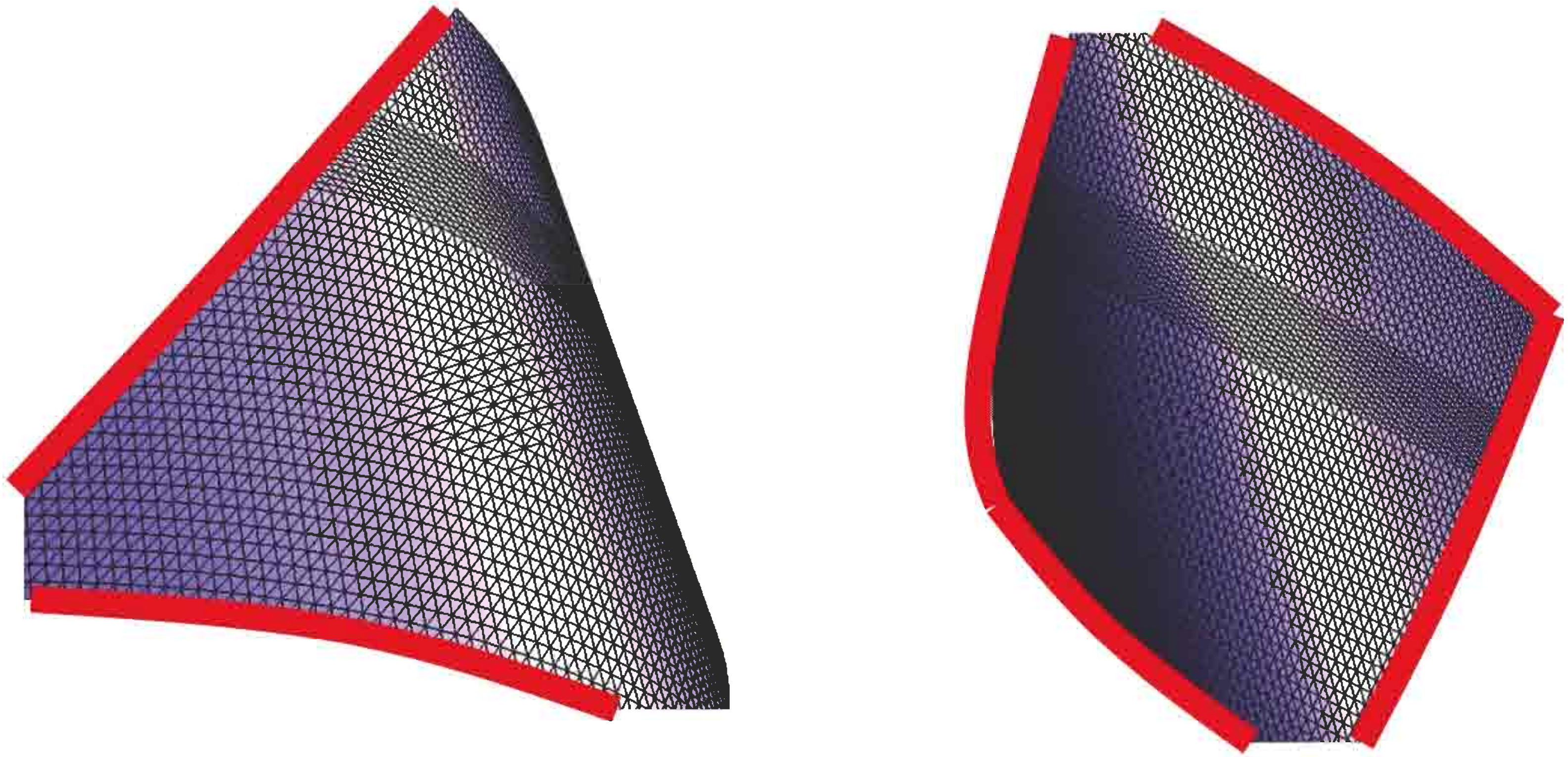


4-8 patch and 1:2 aspect ratio stripe

flat plate, boundary prescribed by quadratic polynomial

interior free to assume minimal-energy

Test problem



4-8 patch and 1:2 aspect ratio stripe

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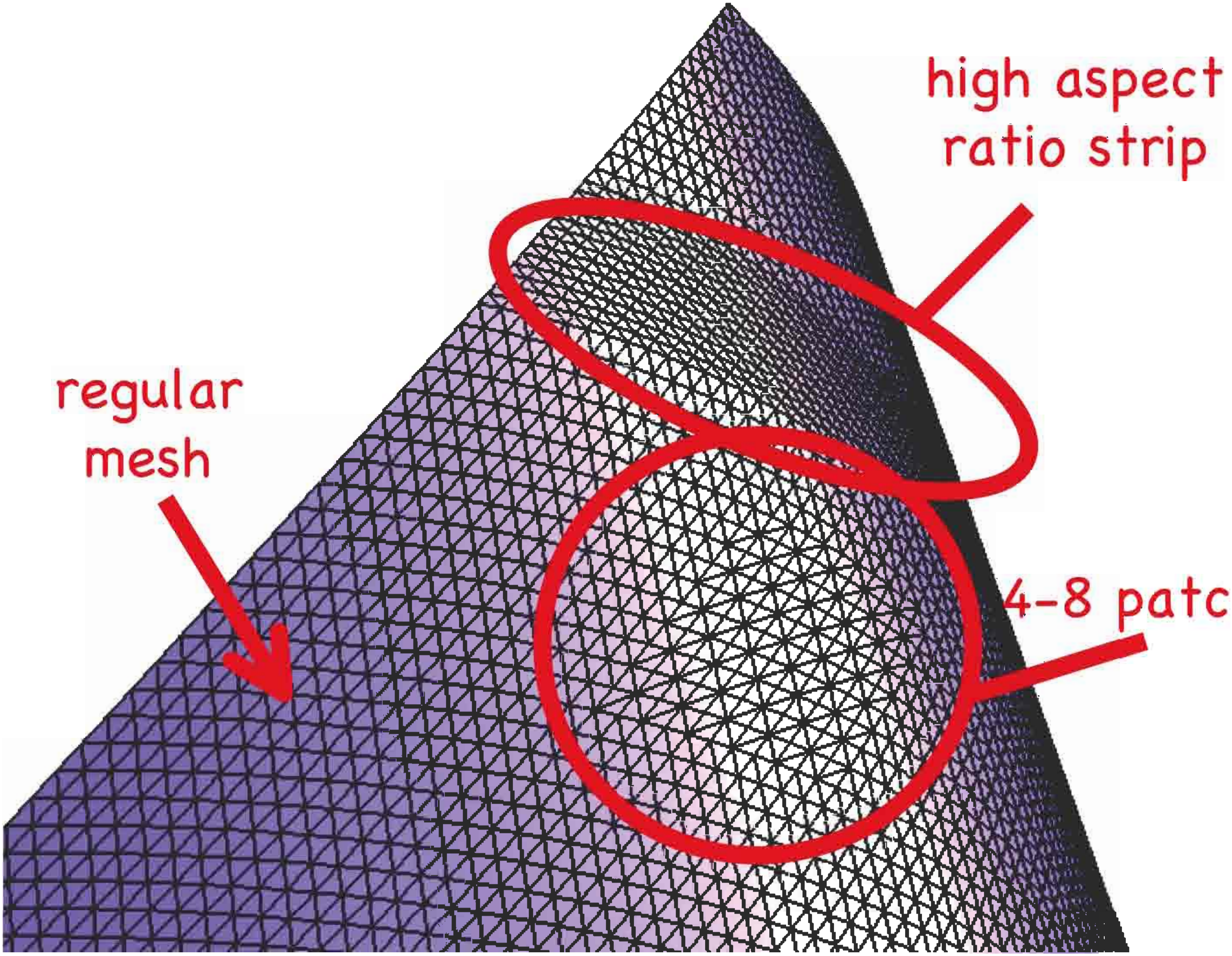
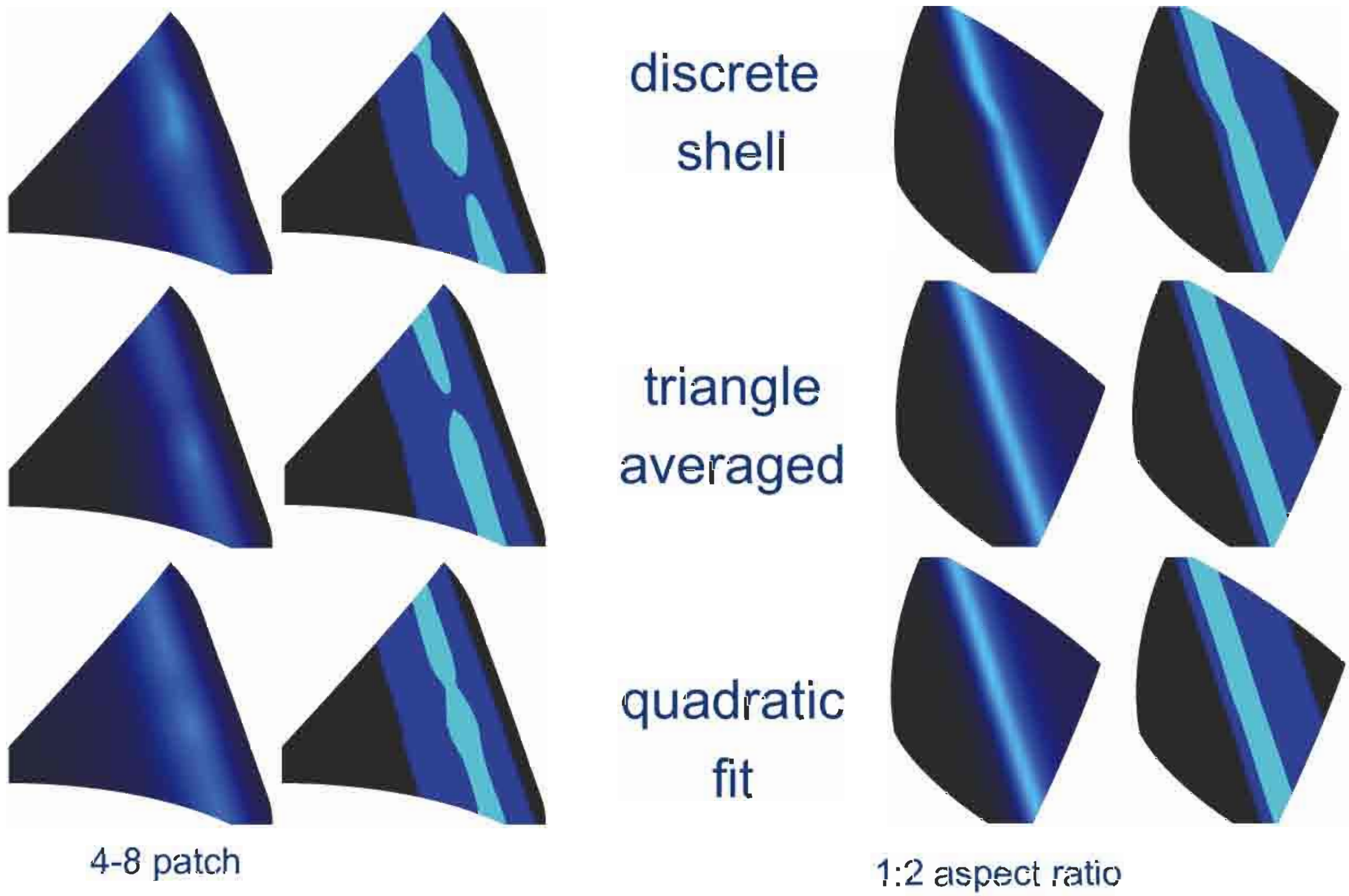
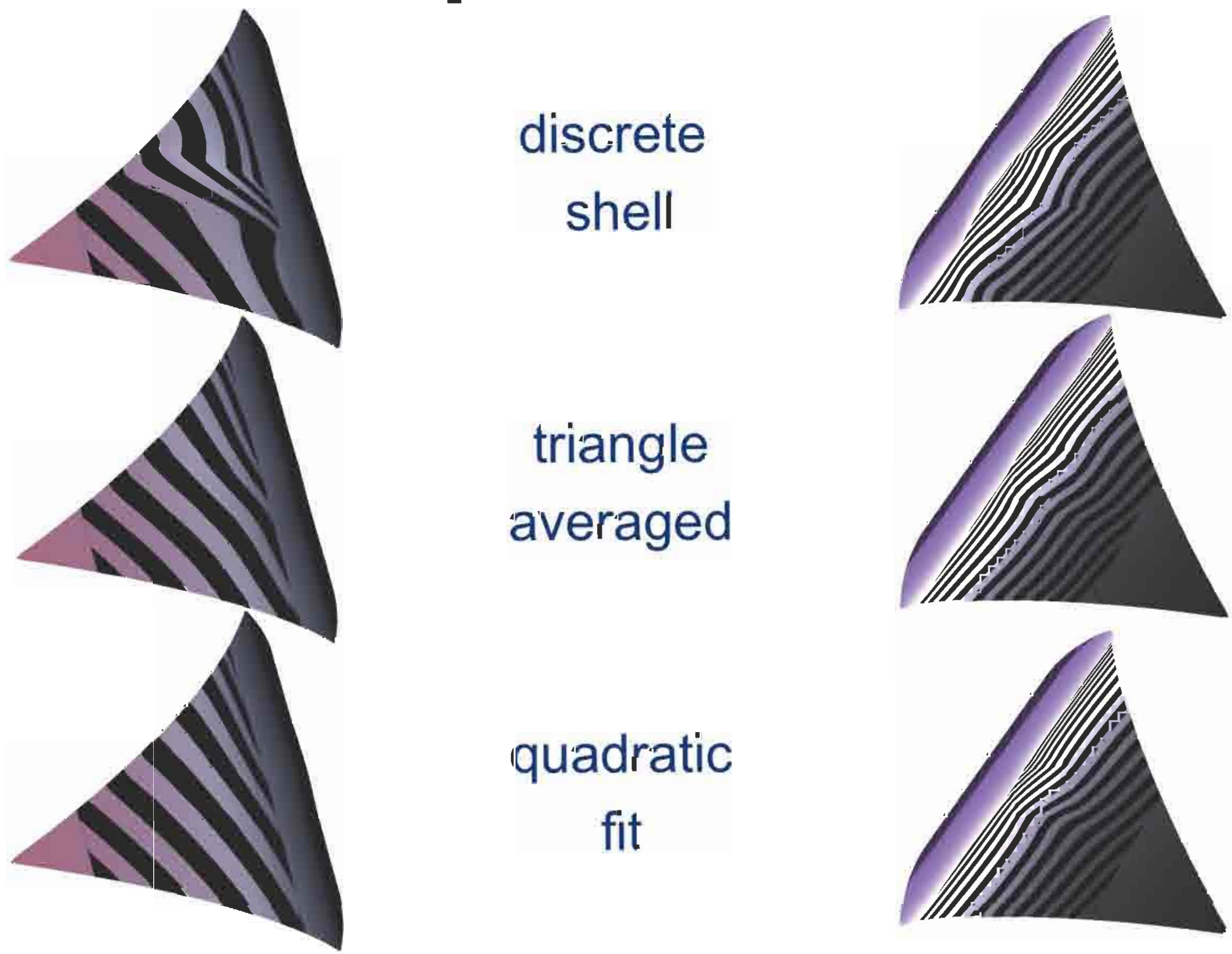

high aspect
ratio strip
regular
mesh
4-8 patc

Shading and highlight maps



Reflection maps



The next step

Need better handle on convergence and mesh-independence

Plus, keep the good stuff:

- geometric invariance:
  - rigid transforms and scaling
- efficiency
  - low FLOPS for gradient and Hessian

Convergence: necessary conditions

Claim

Two necessary conditions must be met so that

for sufficiently fine meshes,
simulation results will be:

- *independent of chosen mesh*
- *in agreement with continuous theory.*

Convergence condition 1

*"The linearized energy of any sampled quadratic polynomial should be reproduced **exactly**."*

- choose any quadratic polynomial surface
- sample the surface
  (even with a coarse mesh)
- measure the linearized energy
- must exactly match the continuous energy

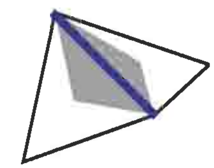
Convergence condition 2

"If the boundary conditions are sampled from a quadratic polynomial, the minimizer of the discrete energy should be ***exactly*** *the sampled polynomial."*

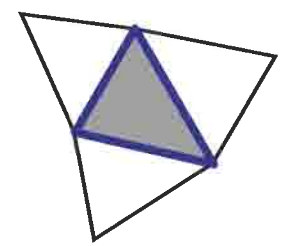
- choose any quadratic polynomial surface
- constrain mesh boundary to samples of poly. surface
- let mesh interior relax to minimum energy
- mesh interior *must* lie exactly on poly. surface

Convergence for general meshes

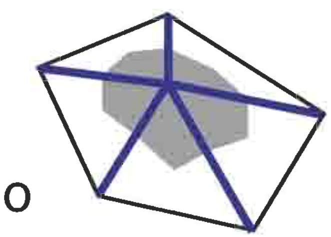
- splines/subdiv. surfaces:
  no quadratic polynomial reproduction near extr. vertices
- discrete shells:
  no quadratic polynomial reproduction



- triangle quadratic fit:
  reproduces quadratic polynomials, but there
  can be lower energy states



- triangle averaged:
  no quadratic polynomial reproduction
- vertex averaged:
  no quadratic polynomial reproduction



- Compare: finite elements only have aspect ratio restrictions

Minimal d.o.f. discretizations

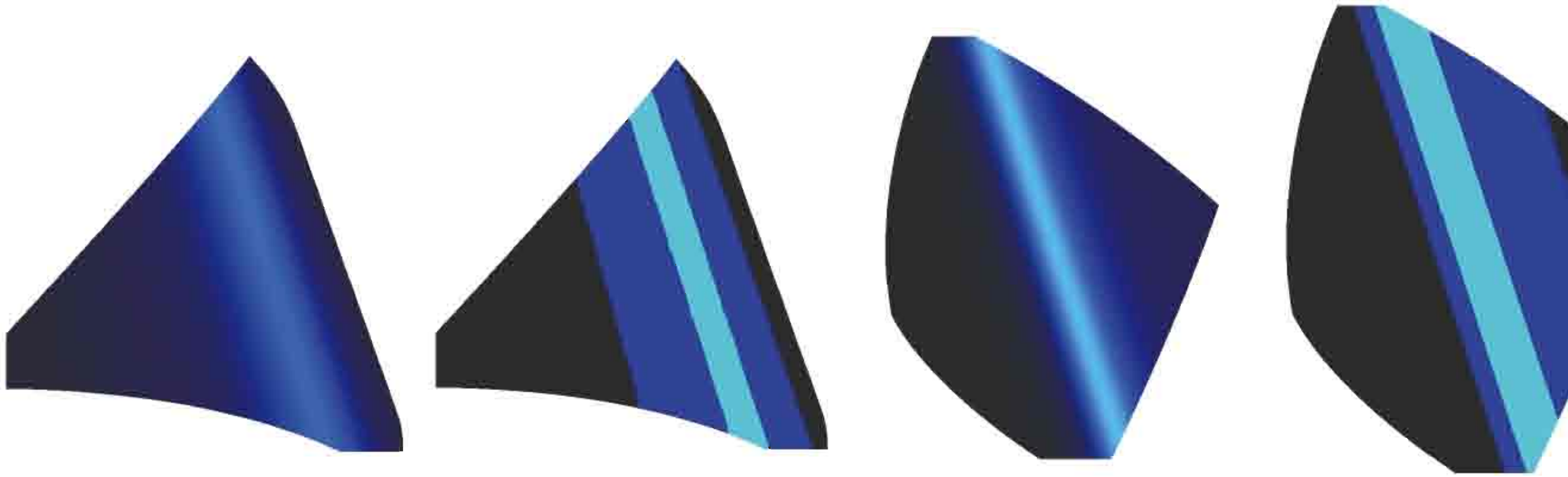
Question:

can we get away with only 6 d.o.f. per energy term?

Answer:

yes, but we must permit d.o.f.s on ***edges****.*

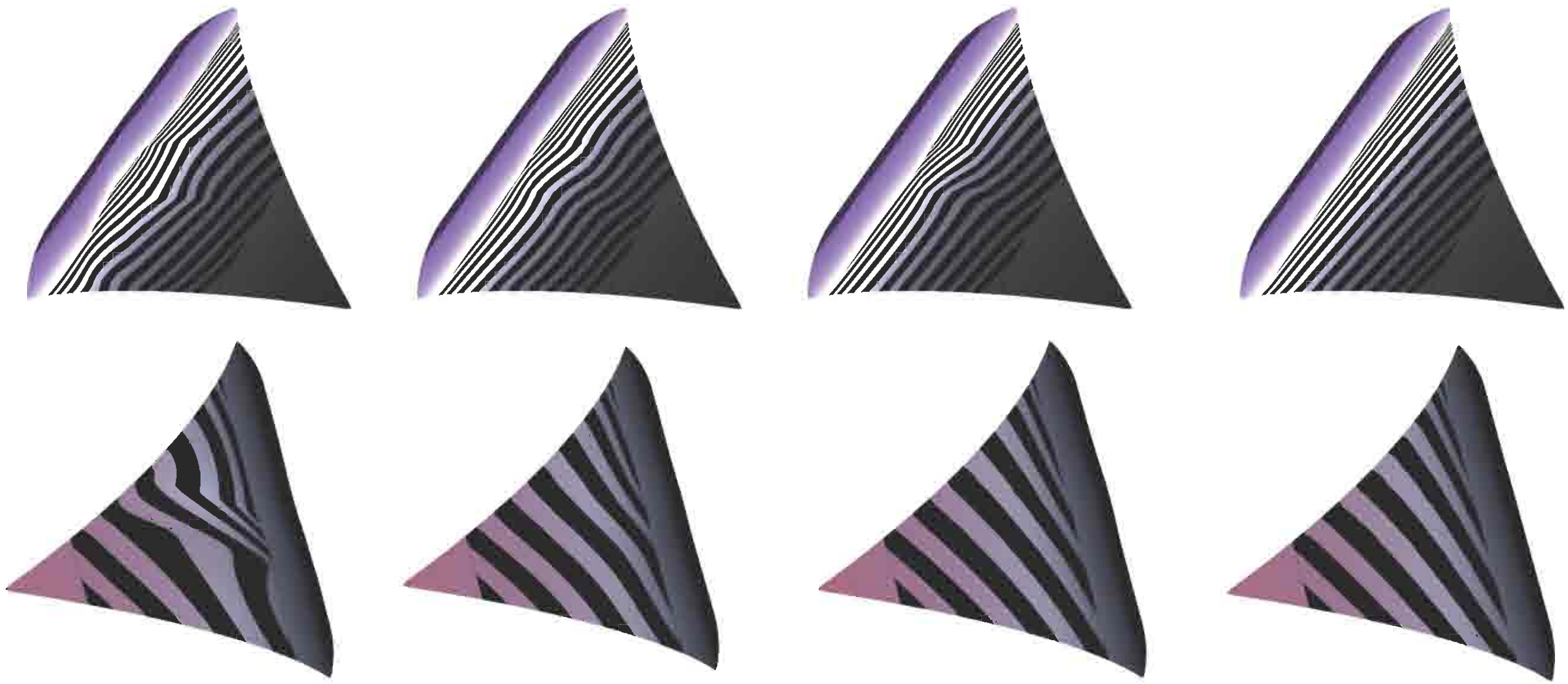
Midedge normal



4-8 patch

1:2 aspect ratio

Reflection maps



discrete shell triangle averaged quadratic fit midedge normal

Mathematics

T. J. Willmore's surfaces

Mathematics

T. J. Willmore's surfaces

$$\frac{1}{4}\int(\kappa_1-\kappa_2)^2 dA = \int(H^2-K)dA$$

Mathematics

T. J. Willmore's surfaces

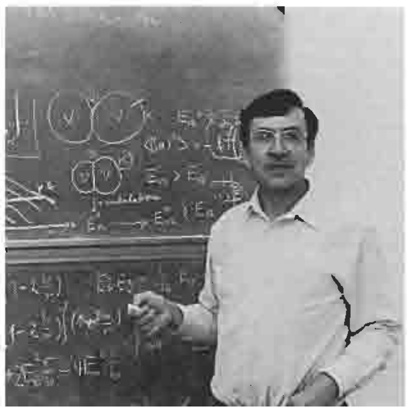
topological invariant

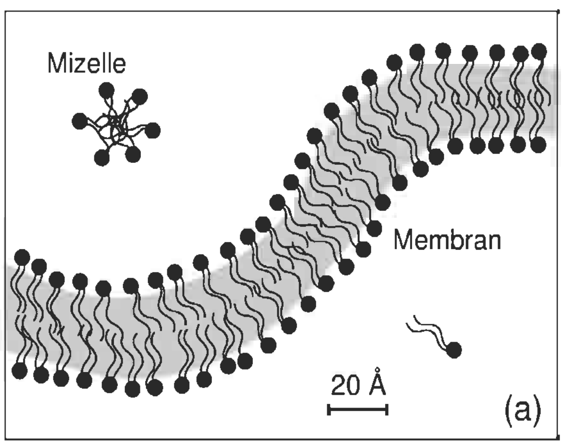
$$\frac{1}{4}\int(\kappa_1 - \kappa_2)^2 dA = \int (H^2 - K)\, dA$$

$$\int H^2 dA = \frac{1}{4}\int(\kappa_1 + \kappa_2)^2 dA$$

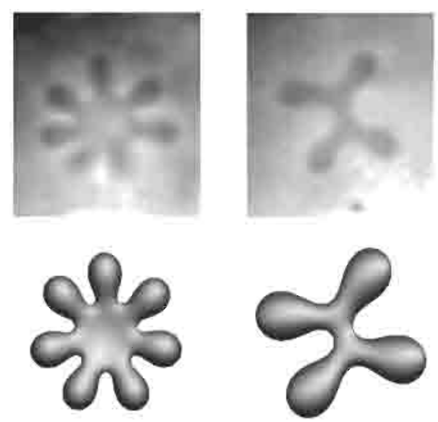
Physics of membranes

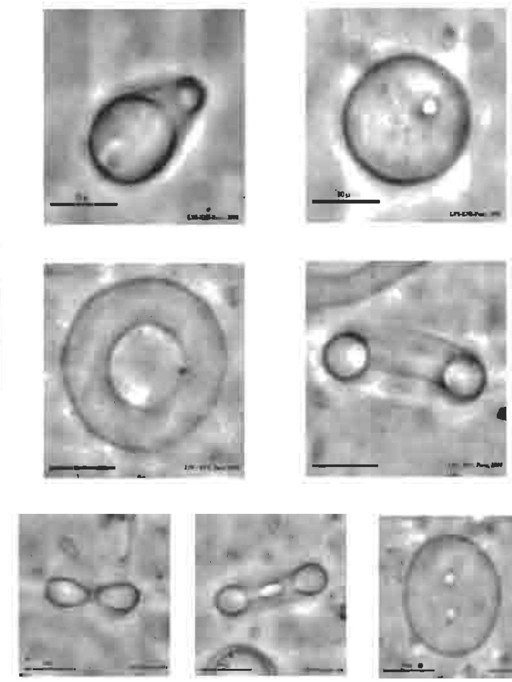
S. Helfrich (FU Berlin)





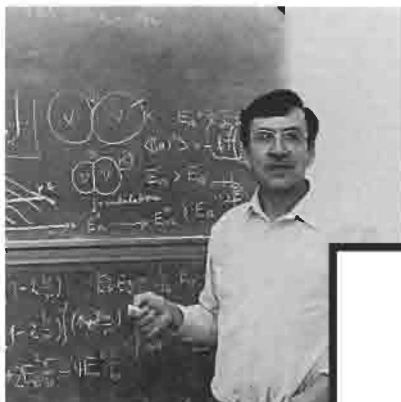
P.B. Canham (U.W. Ontario)

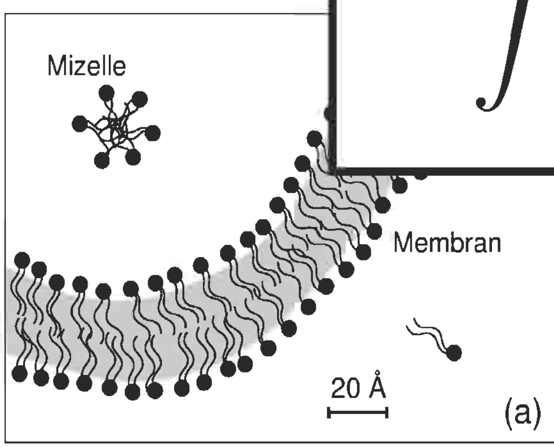




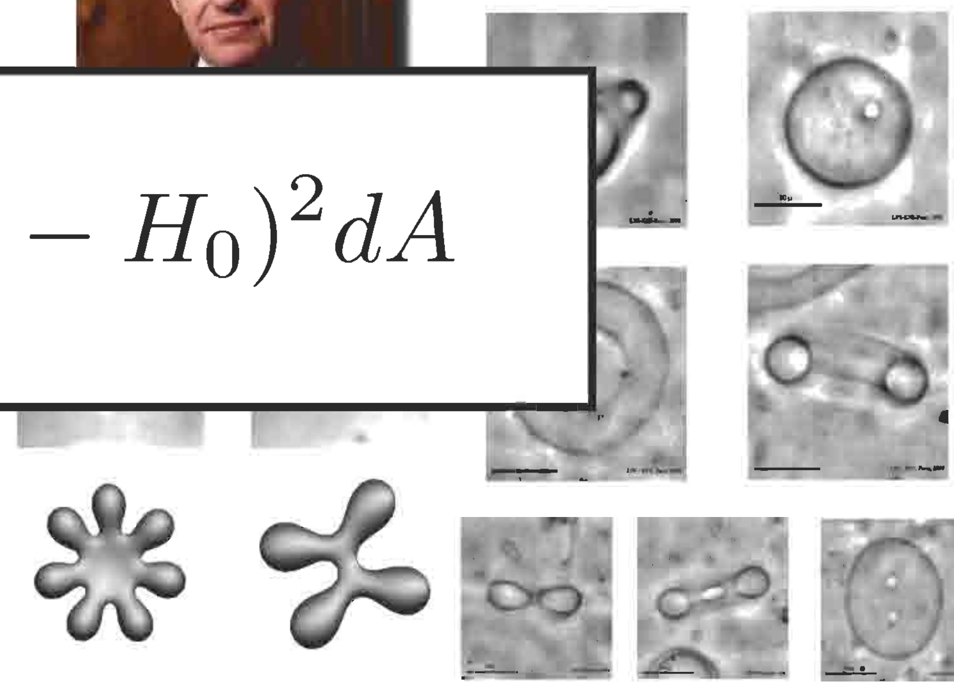
Physics of membranes

S. Helfrich (FU Berlin) P.B. Canham (U.W. Ontario)



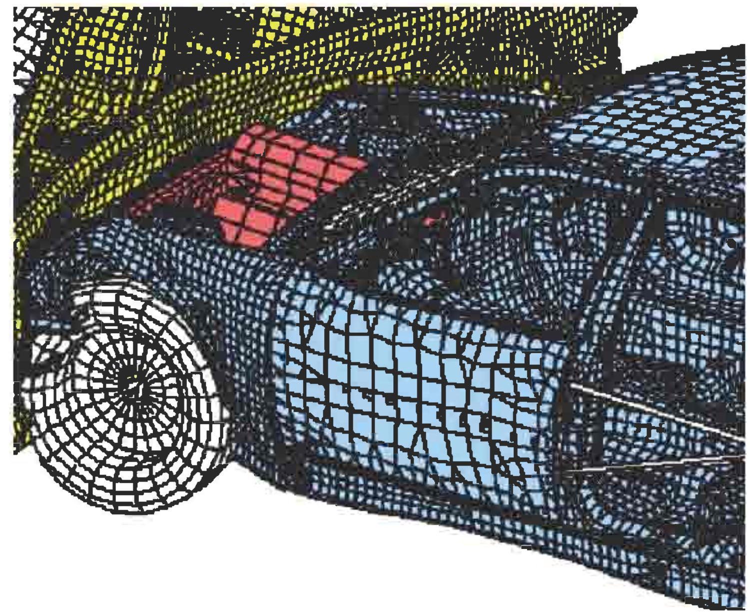


$$\int (H - H_0)^2 dA$$



Engineering

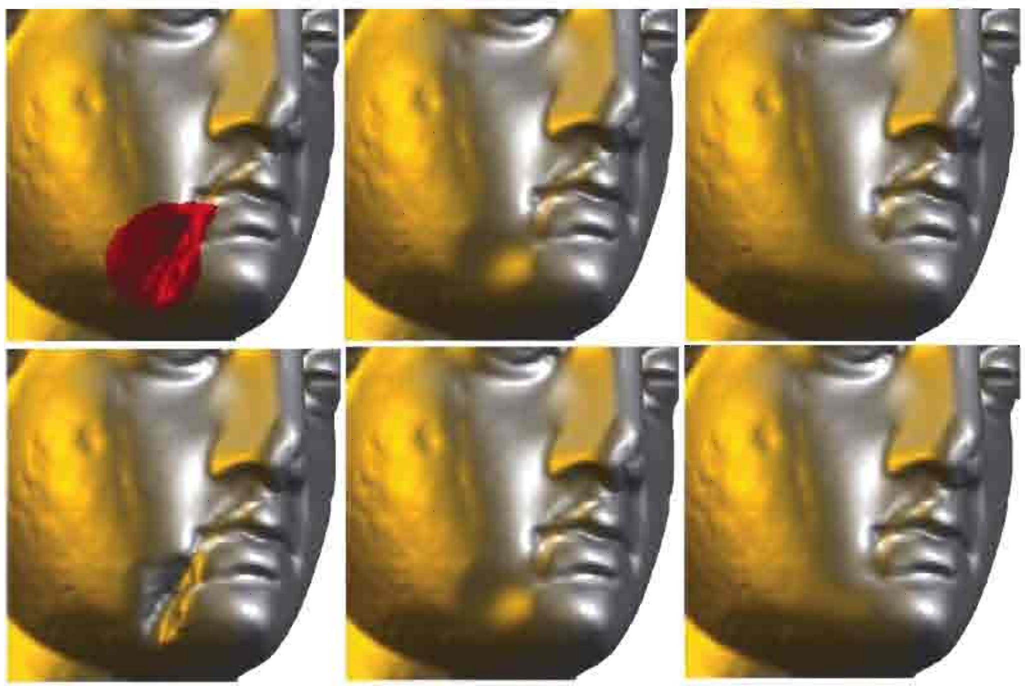
Civil, mechanical, aeronautical design

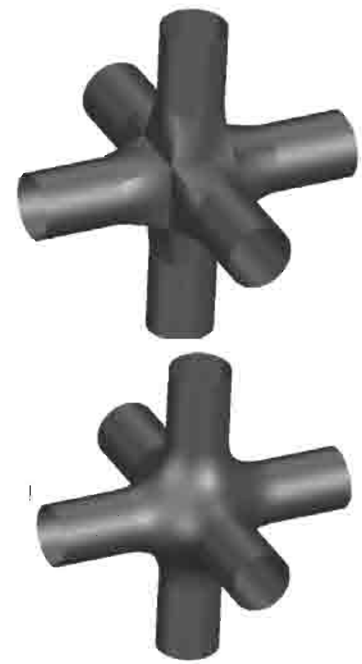


Geometric modeling

Surface fairing and reconstruction

U. Clarenz, U. Diewalda,G. Dziuk, M. Rumpf, R. Rusu (2004)

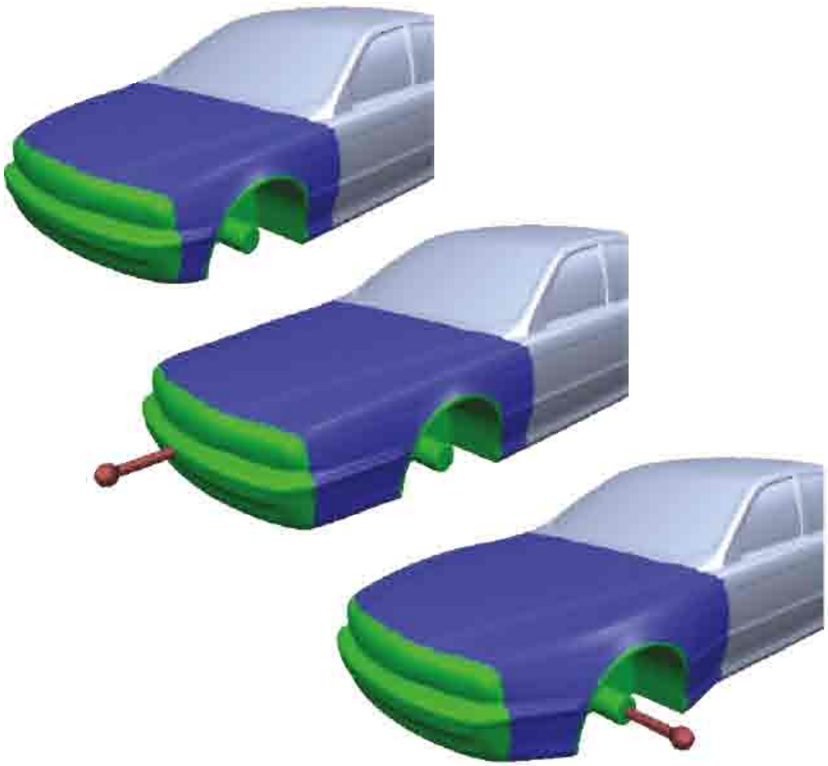




Geometric modeling

Variational modeling

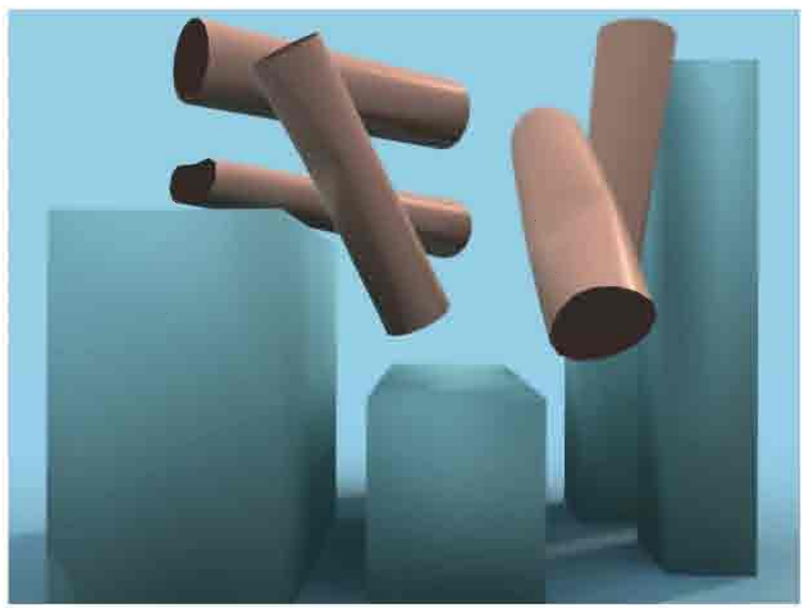
M. Botsch L. Kobbelt

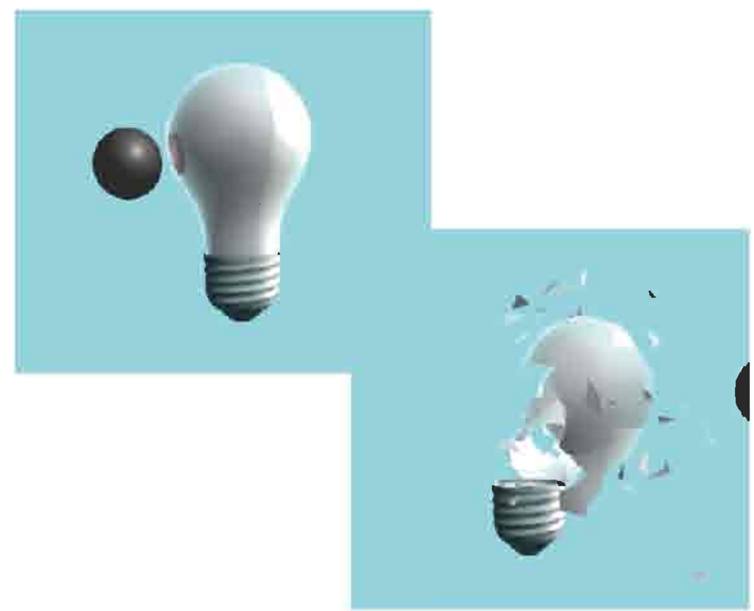


Physically-based animation

Shells: elasticity, plasticity, fracture

with D. Zorin, A. Secord, Y. Gingold, J. Han





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Multi-res Modeling Group, Caltech
Media Research Lab, Courant Institute, NYU