Physical Simulation using Curvature Energies

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Thin shells and thin plates

Thin, flexible objects

Shells are naturally *curved*

Plates are naturally *flat*



Related work

Researchers in the graphics community:

- Terzopoulos, Bridson, Breen, etc.
  - mass-spring and tensorial models for cloth
- Bobenko & Suris, Pai
  - discrete models of elastic curves

[Choi and Ko]

Euler's elastica

Early formulation of elastic curves

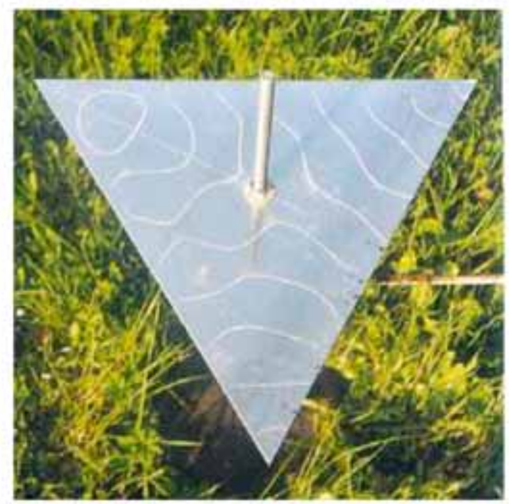
$$E^{\text{bend}} = \int_0^l \kappa(s)^2 ds$$

Bernoulli began generalization to surfaces

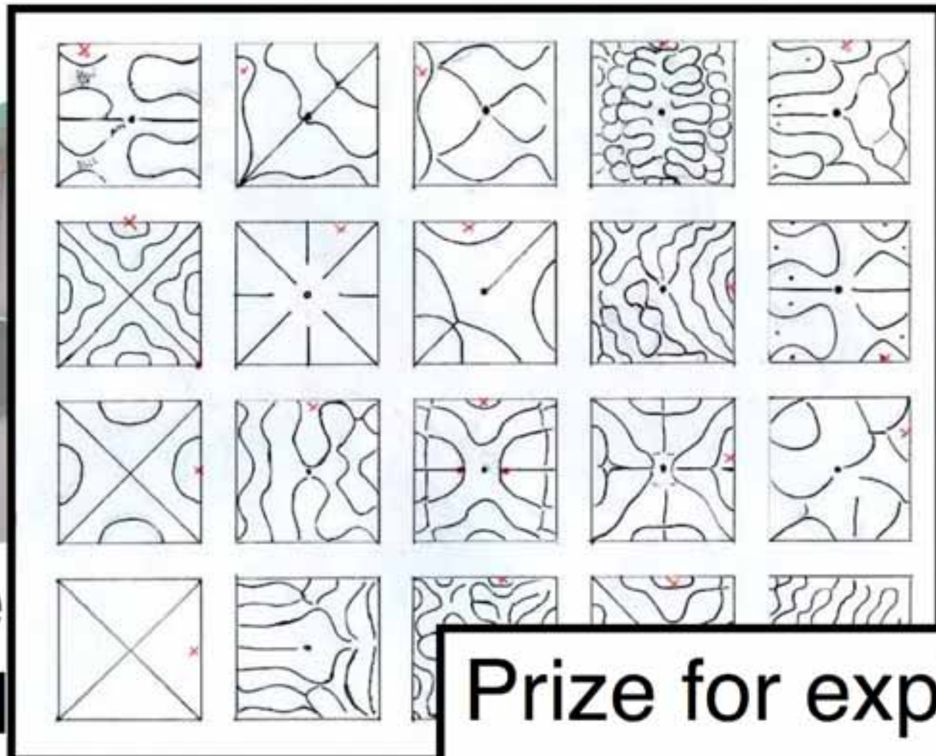
Chladni's vibrating plates

Plate vibrated by violin bow

Sand settles on nodal curves



Chladni's vibrating plates



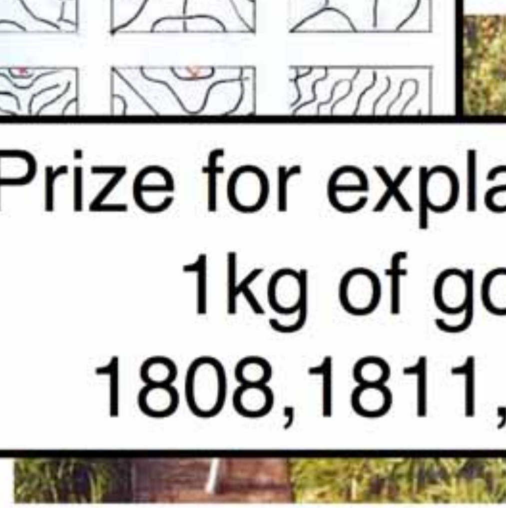


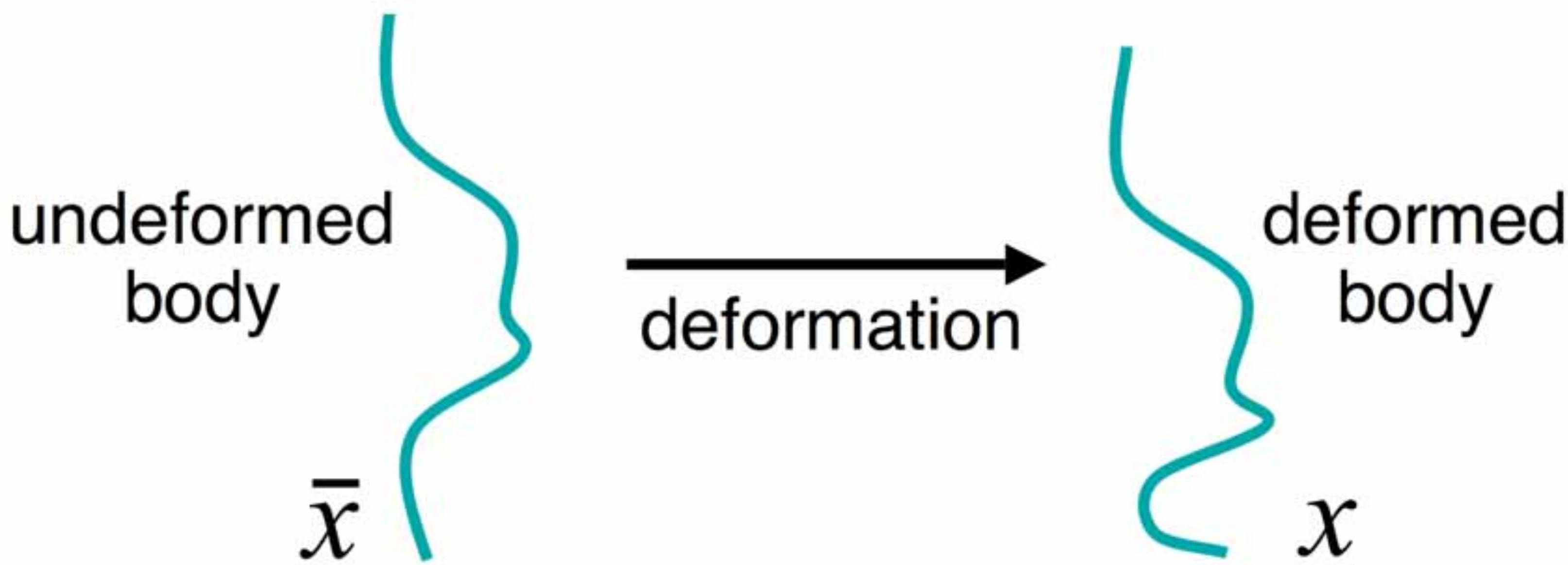
Plate
viol

Sand settles on
nodal curves

Prize for explanation:
1kg of gold,
1808,1811,1815

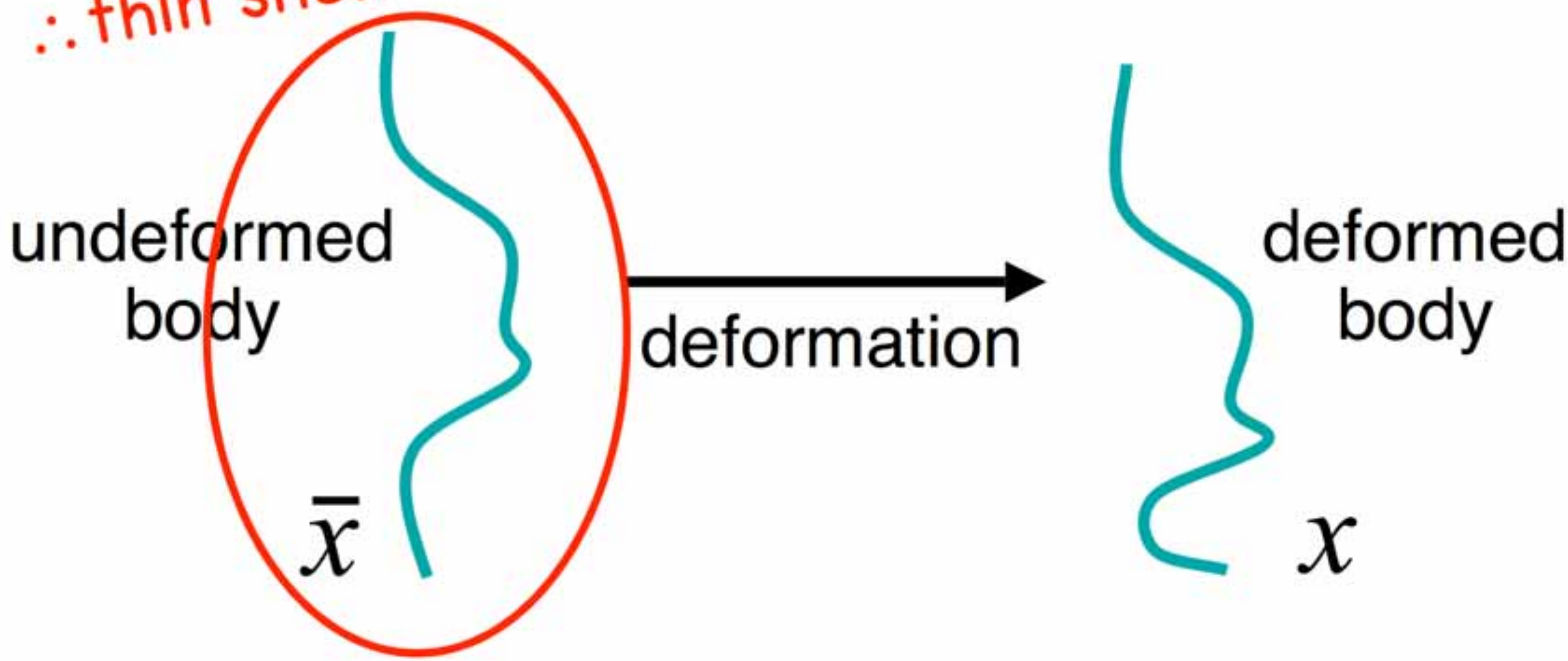
Problem setup

What is the
deformation *energy*?



Problem setup

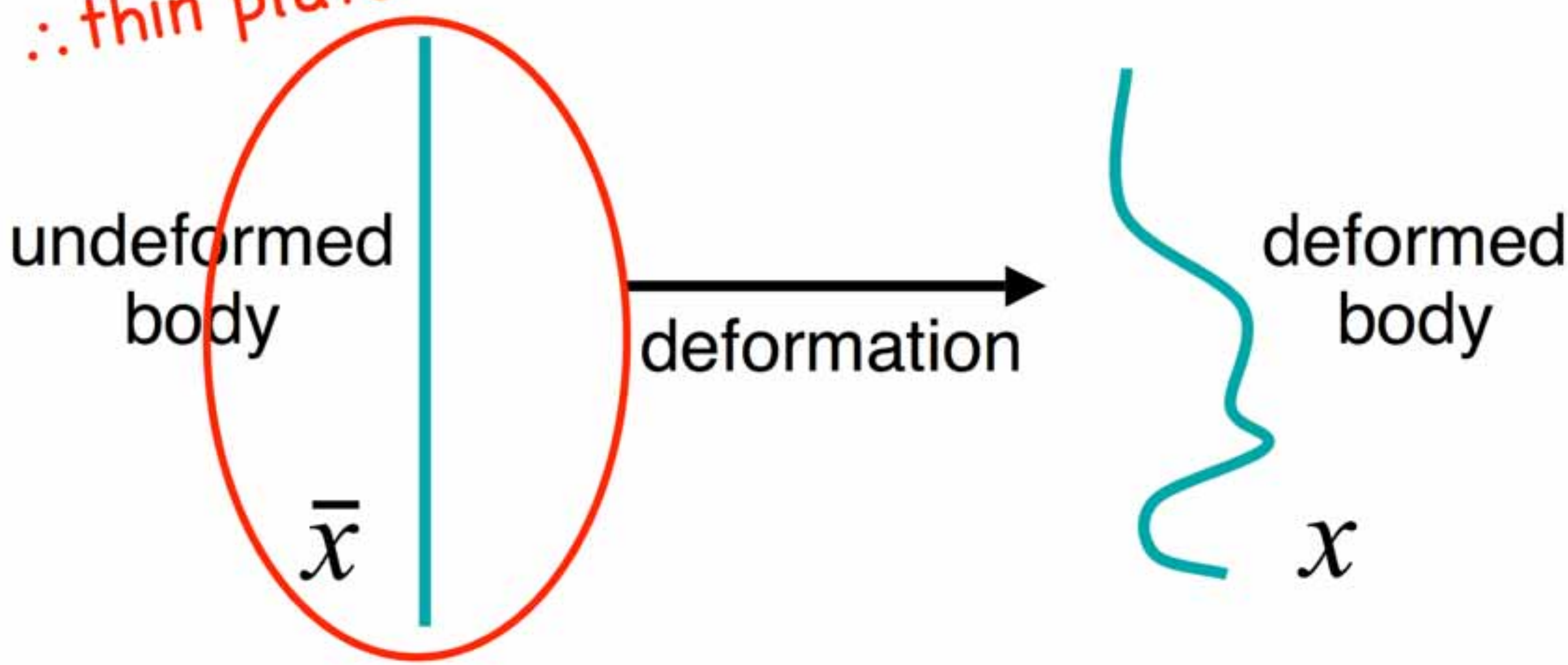
What is the deformation *energy*?



Problem setup

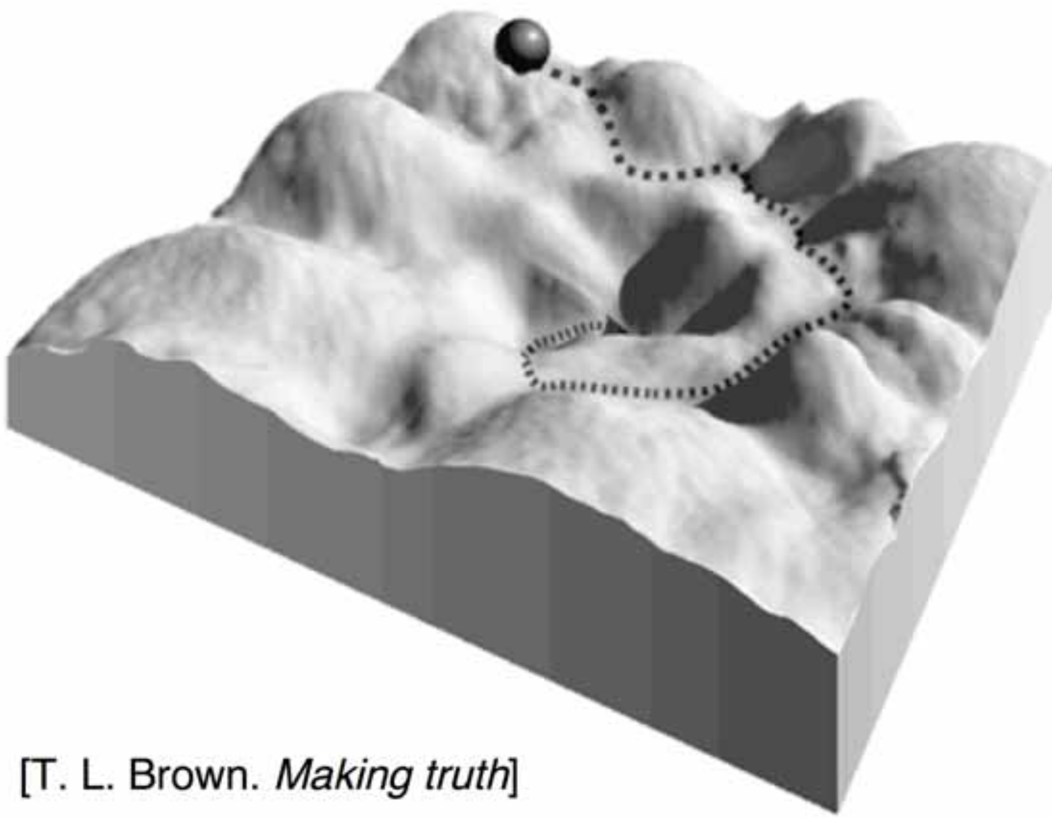
undeformed config.
is flat
∴ thin plate

What is the deformation *energy*?



Problem setup

Energy is a non-negative scalar function

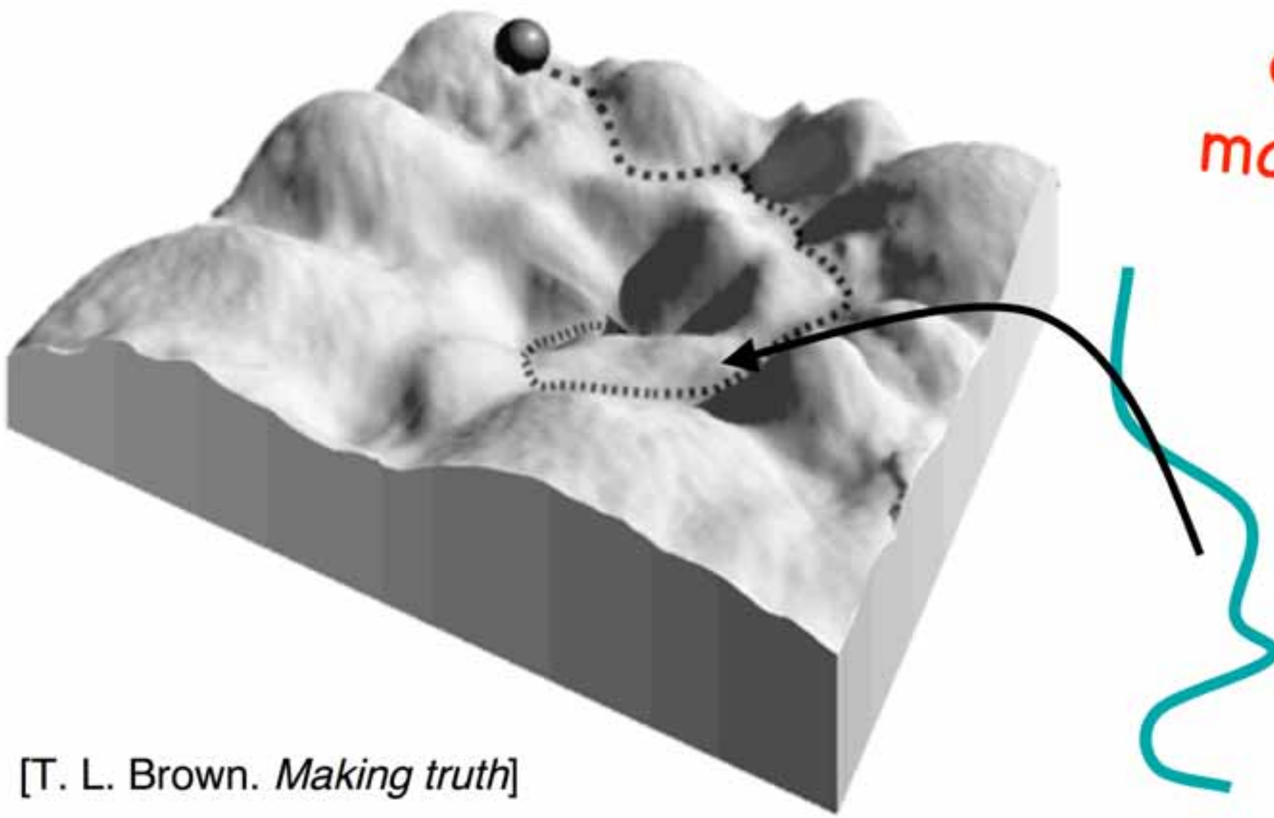


[T. L. Brown. *Making truth*]

Problem setup

real number,
coordinate-frame invariant

Energy is a non-negative scalar function



each config.
maps to a point
on energy
landscape

[T. L. Brown. *Making truth*]

Problem setup

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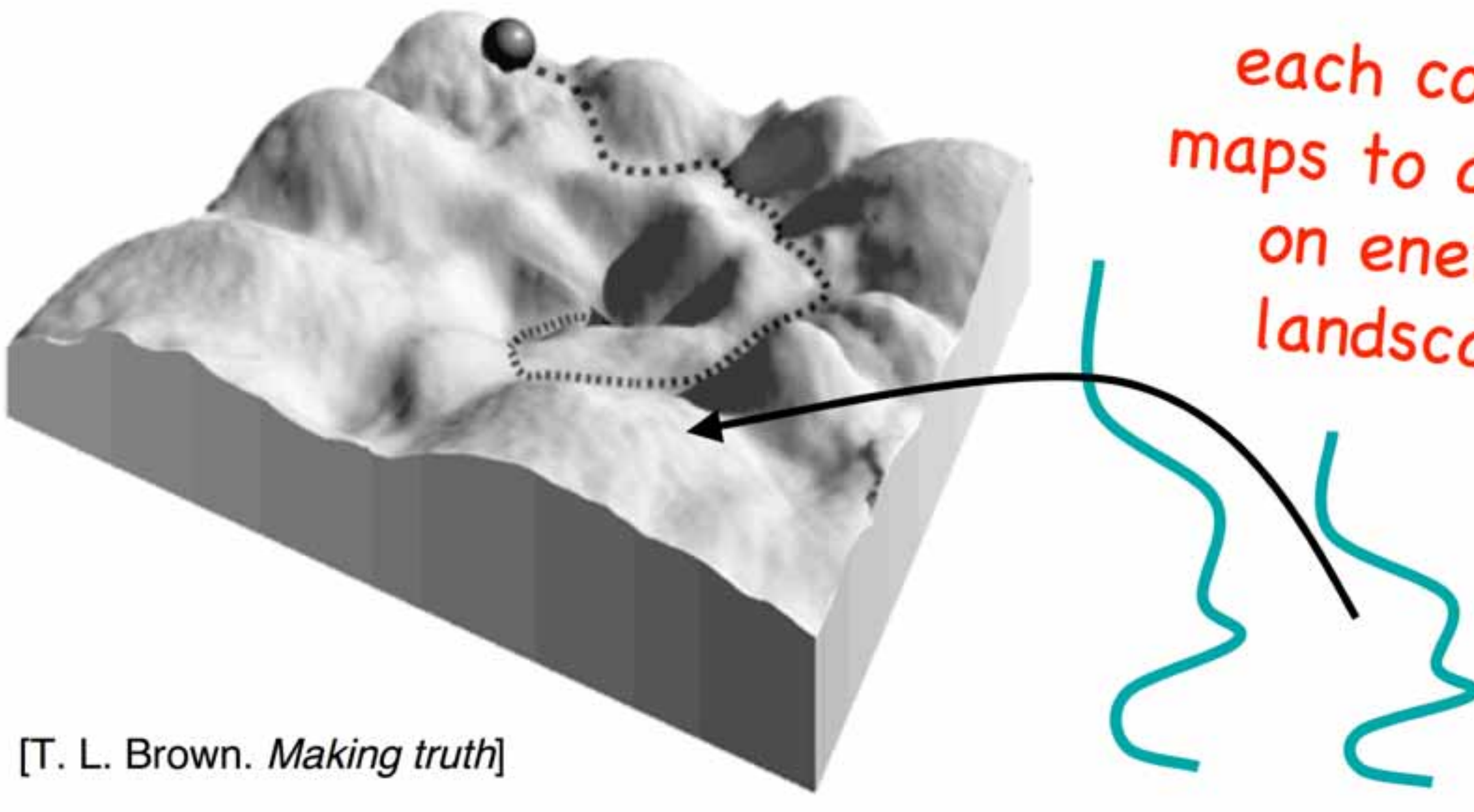
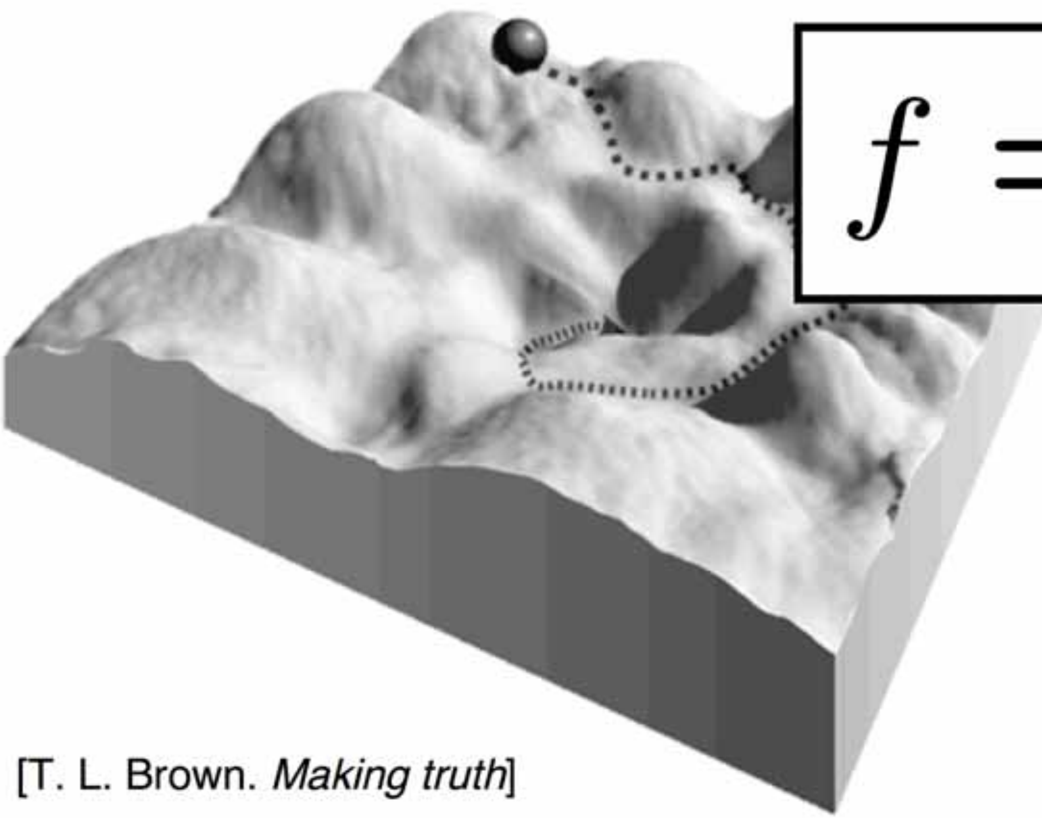


[T. L. Brown. *Making truth*]

Problem setup

Internal forces push "downhill"



$$f = -\nabla E$$

[T. L. Brown. *Making truth*]

Plates

Germain

Poisson

Navier

Thin plate energy

Germain's argument:

- bending energy must be a symmetric even function of principal curvatures

Thin plate energy

Germain's argument:

- bending energy must be a symmetric even function of principal curvatures

$$E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4}\int (\kappa_1 + \kappa_2)^2 dA$$
$$= \int H^2 dA$$

Thin plate energy

Poisson's linearization

- assuming small displacements, approximate curvature by second derivatives

$$E^{bend} = f(\kappa_1, \kappa_2) = \frac{1}{4} \int (\kappa_1 + \kappa_2)^2 dA$$

$$E^{bend}_{lin} = \int (\Delta f)^2 dA$$

Thin plate energy

Navier's equation

- to find minimizer for linearized energy, solve a partial differential eqn (PDE)

$$\Delta^2 f = 0$$

$$E_{lin}^{bend} = \int (\Delta f)^2 dA$$

Thin plate energy

Navier's equation

- to find minimizer for linearized energy
  sol

$$\partial_{uuuu} f + 2\partial_{uuvv} f + \partial_{vvvv} f$$

$$\Delta^2 f = 0$$

$$E_{lin}^{bend} = \int (\Delta f)^2 dA$$

Axiomatic approach

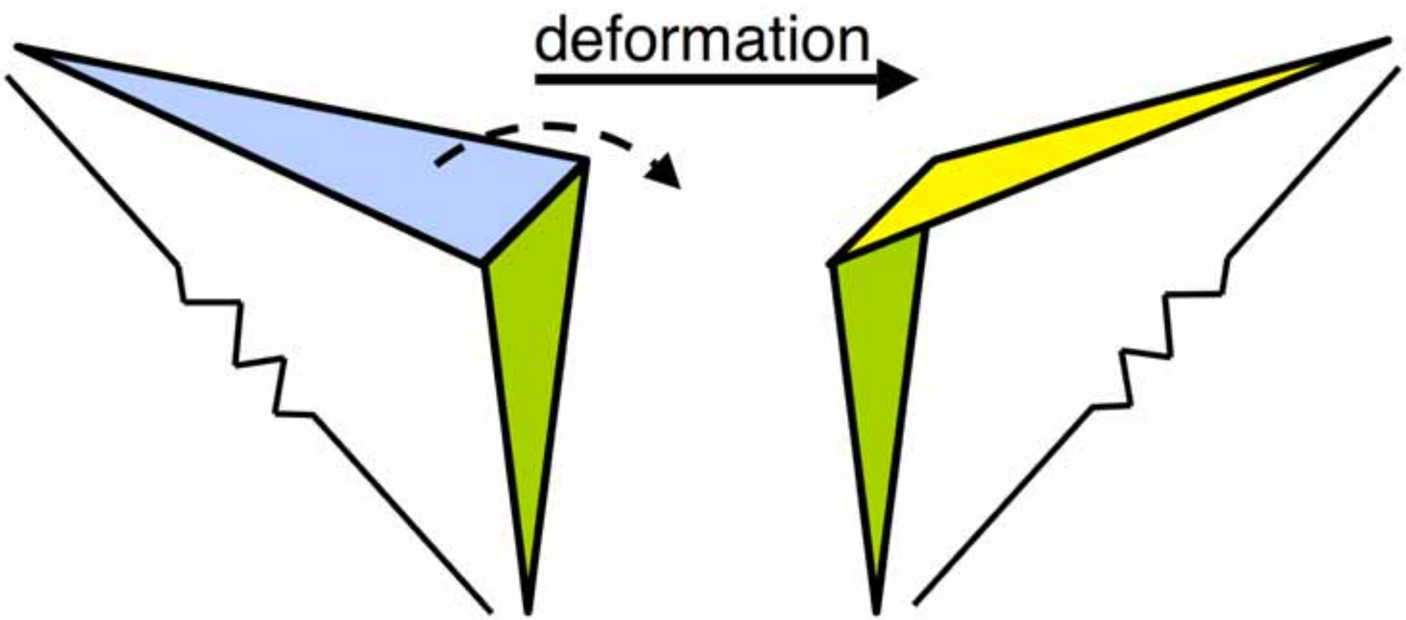
Energy should be:

- symmetric even func’n of principal curvatures
- extrinsic measure
- smooth w.r.t. change in shape
- invariant under rigid-body motion
- simple to compute
- easy to understand

What about masses and springs?

Diagonal springs don't work for shells.

- undeformed configuration is *curved*
- incorrect energy minima



Axiomatic "discrete shells"

"Simplest" answer to desiderata

$$\boxed{(H - H_0)^2}$$

Derivation:

extrinsic change in *shape* operator

$$[\mathrm{Tr}(\varphi^* \mathsf{S}) - \mathrm{Tr}(\bar{\mathsf{S}})]^2$$

Computing discrete shells

Elastic energy = $\frac{K_B}{2} \sum_i (\theta_i - \bar{\theta}_i)^2 \frac{\|\bar{e}_i\|}{\bar{h}_i}$